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Dissociation between magnitude comparison and relation identification across different formats for rational numbers

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ABSTRACT
The present study examined whether a dissociation among formats for rational numbers (fractions, decimals, and percentages) can be obtained in tasks that require comparing a number to a non-symbolic quantity (discrete or else continuous). In Experiment 1, college students saw a discrete or else continuous image followed by a rational number, and had to decide which was numerically larger. In Experiment 2, participants saw the same displays but had to make a judgment about the type of ratio represented by the number. The magnitude task was performed more quickly using decimals (for both quantity types), whereas the relation task was performed more accurately with fractions (but only when the image showed discrete entities). The pattern observed for percentages was very similar to that for decimals. A dissociation between magnitude comparison and relational processing with rational numbers can be obtained when a symbolic number must be compared to a non-symbolic display.

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Doubtless, the most salient property of numbers is that they convey magnitudes. Numerous studies of numerical magnitude comparisons have yielded a *symbolic distance effect*: comparisons of numbers that are closer in magnitude (e.g., 7 vs. 8) are slower and more error prone than comparisons of numbers that are farther apart (e.g., 2 vs. 8; Moyer & Landauer, 1967). A similar distance effect is observed in children (e.g., Brannon, 2002). A major goal of recent research in cognitive psychology and education has been to find ways to aid in learning the magnitudes of rational numbers, such as fractions (e.g., Bailey et al., 2015; Booth, Newton, & Twiss-Garrity, 2014; Vukovic et al., 2014). In this work, conceptual understanding of fractions is often identified with an understanding of fraction magnitudes (e.g., 4/5 is larger than 1/2), the

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mathematical principles relevant to fractions (e.g., an infinite number of fractions can be placed between any two other fractions), and notations for expressing equivalent fraction magnitudes (e.g., 3/4 = 6/8 = 0.75) (Bailey et al., 2015; Siegler & Lorti-Forgues, 2015).

Fractions and decimals are often viewed simply as alternative notations, which (other than rounding error) are equivalent in magnitude (e.g., 3/8 vs. 0.375). Often overlooked is the fact that, in addition to representing magnitudes, the core definition of a fraction (i.e., a rational number that can be expressed as the quotient of two natural numbers, a/b, where $b \neq 0$) is inherently *relational*. Specifically, the bipartite structure of the a/b notation expresses a relational model for the quantities corresponding to the numerator and denominator (see Figure 1). The form a/b expresses a division relation between natural numbers, which has the form of a fraction; whereas the output of the division, c, expresses the magnitude of that relation and can be a decimal (with magnitude less than 1 when $a < b$). As illustrated in Figure 1, the bipartite structure of the fraction provides a direct mapping to countable subsets in a visual display (e.g., a subset of three white objects within a set of eight). In contrast, the decimal equivalent loses the two-dimensional structure of the fraction. Instead, it provides a one-dimensional measure of a portion of a continuous unit quantity. (For discussions of the relationship between dimensionality of conceptual structures and relational complexity, see Halford, Wilson, & Phillips, 1998, 2010.)

This analysis suggests that alternative notations for numbers may differ in their inherent effectiveness for performing tasks that tap into magnitude versus relational knowledge.

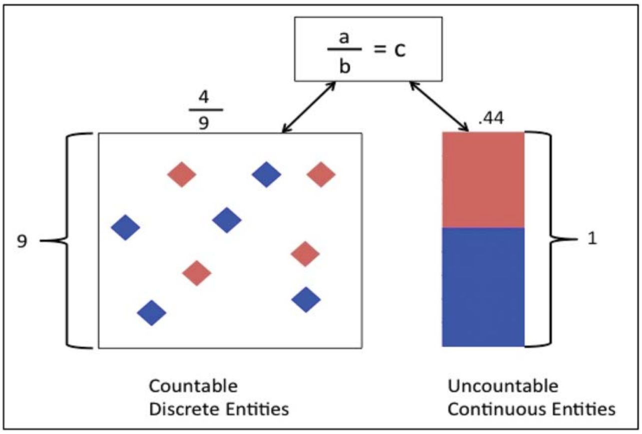

Figure 1. The bipartite structure of a fraction maps to countable subsets in a visual display (left), whereas the decimal equivalent represents a one-dimensional magnitude (right). Reprinted with permission from Rapp et al. (2015).

In particular, the different formats for rational numbers appear to be selectively aligned with discrete versus continuous quantities: fractions with the former, decimals with the latter (Rapp, Bassok, DeWolf, & Holyoak, 2015). The contrast between discreteness and continuity, which is closely linked with the linguistic distinction between count and mass nouns (e.g., marbles vs. water, respectively; Bloom & Wynn, 1997), is a basic ontological distinction that affects how people parse the world. For example, Spelke, Breinlinger, Macomber, and Jacobson (1992) argued that young babies use this distinction to discriminate between objects: continuity of motion indicates that a single object is moving in space, whereas discontinuity indicates the existence of more than one object.

Recent research has shown that differences in the formats for expressing rational numbers influence how people reason with them in different contexts. When people are asked to make speeded magnitude comparisons between pairs of numbers, integers and decimals show an advantage in speed and accuracy relative to fractions (e.g., DeWolf, Chiang, Bassok, Holyoak, & Monti, 2016; DeWolf, Grounds, Bassok, & Holyoak, 2014; Lee, DeWolf, Bassok, & Holyoak, 2016; Schneider & Siegler, 2010). The magnitude comparison task requires processing one-dimensional representations of magnitude, which is more natural for integers and decimals than for fractions. By contrast, in reasoning contexts that evoke relational processing, fractions show an advantage relative to decimals when used to characterise relations between sets of discrete and countable entities (DeWolf, Bassok, & Holyoak, 2015a; Lee et al., 2016; Plummer, DeWolf, Bassok, Gordon, & Holyoak, 2017; for a review, see DeWolf, Bassok, & Holyoak, 2017).

Goals of the present study

In the present paper, we report two experiments that aim to more directly demonstrate a dissociation between tasks requiring magnitude comparison (Experiment 1) and relational reasoning (Experiment 2) for different rational number formats.

The present experiments extend previous studies in three ways. First, almost all previous studies of magnitude comparisons have involved comparisons either of two symbolic magnitudes (i.e., numbers) or else two non-symbolic magnitudes (i.e., pictured quantities). However, a primary purpose of using a symbolic number is to relate the number to quantitative properties of non-symbolic entities. A small number of studies have examined how holistic processing of rational number magnitudes may vary depending on the symbolic or non-symbolic nature of the representation. For example, Matthews and Chesney (2015) demonstrated a distance effect when adults performed magnitude comparisons of symbolic ratios (fractions) with non-symbolic ratios (ratios of quantities composed of either discrete dots or continuous circles; see also Matthews & Lewis, 2016). However, the present Experiment 1 is the first to investigate magnitude comparisons when both the type of non-symbolic quantity (continuous vs. discrete) and the type of symbolic number format (fraction, decimal, percentage) are systematically varied. If magnitudes are most naturally represented on a continuous mental number line, then comparisons will be made more easily when the non-symbolic quantity is continuous. If such a continuous advantage holds even for comparisons with fractions, this would suggest that a fraction is converted into a continuous magnitude in order to compare it with a non-symbolic quantity.

A second and related goal was to eliminate possible confoundings between the type of decision required and other variables. Previous studies showing a decimal advantage in magnitude comparisons have all involved numbers only, whereas studies showing a fraction advantage in relational processing have all involved a comparison of a number to a non-symbolic display. Moreover, the range of numbers involved has not been controlled across different tasks. These confounds limit the interpretation of prior findings. In the present study, both magnitude comparisons (Experiment 1) and relation judgements (Experiment 2) required application of a symbolic number to a non-symbolic display (a picture, which could show either a continuous quantity or discrete entities). In addition, we matched the actual numbers used in the two tasks (thus eliminating any possible differences in performance attributable to the specific numbers, as opposed to the type of judgement required).

Third, we expanded the formats to be compared by using three notations for rational numbers: fractions (e.g., $^3/_4$), decimals (e.g., 0.75) and percentages (e.g., 75%). Whereas fractions and decimals have received considerable attention, no studies (to our knowledge) have examined the impact of the percentage format as well. More generally, the paucity of research on knowledge of percentages has been highlighted in a recent review (Tian & Siegler, 2017). Different views of percentages lead to alternative predictions about how they will compare with fractions and decimals in performing different numerical tasks. According to Parker and Leinhardt (1995), percentages are "privileged proportions", and their most common interpretation and use is to represent fractions with the denominator of 100 (e.g., 75/100 = 75%). That is, the percentage notation (%) indicates that the number, like a fraction, represents a proportional relation. However, these theorists view percentages as "privileged proportions" in that they "take advantage of the natural and powerful ordering of the decimal numerations system" (p. 445). That is, the format of percentages is metric (base 10) and one-dimensional, much like decimals. Accordingly, the pattern of performance to be expected for percentages relative to fractions and decimals is unclear. The underlying proportional meaning of percentages may affect performance in a manner similar to the fraction format, but their metric notation may influence performance in a manner similar to decimals.

We report two studies that investigated adults' ability to perform magnitude comparisons and relational identifications across symbolic and non-symbolic magnitudes. In Experiment 1, participants compared the magnitude of a proportion conveyed by a picture (continuous or discrete) to that of a rational number in the format of a decimal, fraction or else percentage. Based on previous research (e.g., DeWolf et al., 2014), we expected performance to be facilitated by decimals relative to fractions. The impact of format might interact with entity type (e.g., better performance for fractions with discrete pictures but for decimals with continuous pictures). However, if magnitude comparisons are naturally based on a continuous internal representation, then discrete pictures may be recoded as continuous quantities (a process that would take extra time), so that comparisons would be easier with decimals for both display types.

In Experiment 2, participants used fractions, decimals and percentages to identify ratio relationships among discrete and continuous entities. Based on previous research (e.g., DeWolf et al., 2015a), we expected that this task would be facilitated by fractions relative to decimals, but only for discrete (i.e., countable) displays. For both tasks, the expected performance for percentages is open to alternative predictions. Percentages may behave like fractions, like decimals, or perhaps in some intermediate fashion.

Experiment 1

Method

Participants

Participants were 44 undergraduate students (mean age = 20.3, 31 female) from the University of California, Los Angeles (UCLA), who received course credit for participating.

Materials and design

The experiment was a 3 (number type: fraction, decimal, percentage) × 2 (entity type: continuous vs. discrete) within-subjects design.

The stimulus on each trial consisted of a picture that was either continuous or discrete (see Figure 2), and a number represented as either a fraction, decimal or percentage. Participants were required to decide if the non-symbolic proportion in the picture was larger or smaller than the number presented. For half of the trials, the number was larger than the proportion shown in the picture. Table 1 lists the stimuli used across the trials. In order to make it easier for participants to process the pictures and numbers, and to highlight the difference between discrete and continuous pictures, the proportions were

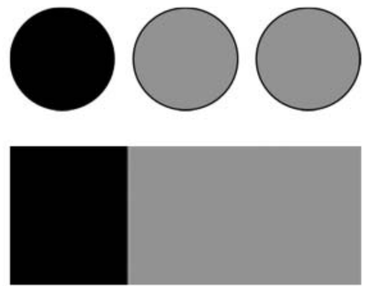

Figure 2. Examples of discrete and continuous stimuli. Both stimuli represent the proportion 1/3, where the black portion represents the numerator value and the grey portion represents the denominator value.

created using only single-digit numbers in the numerator and denominator positions. Each magnitude was matched across number types, such that fractions were divided and rounded to two places for equivalent decimals and multiplied by 100 for equivalent percentages (e.g. $^3/_4$, 0.75, 75%). There were a total of 20 trials for each combination of picture type and number type (i.e., 120 trials in total).

Procedure

Participants were shown the stimuli randomised within number-type blocks (40 trials in each block). Within each block, the order of trials was randomised

Table 1. Stimuli used for magnitude comparison task (Experiment 1).

Non-symbolic proportion	Symbolic decimal	Symbolic fraction	Symbolic percentage (%)
8/9	0.75	3/4	75
7/8	0.80	4/5	80
5/8	0.60	3/5	60
5/7	0.88	7/8	88
5/6	0.78	7/9	78
4/7	0.67	2/3	67
4/5	0.11	1/9	11
3/8	0.20	1/5	20
3/7	0.22	2/9	22
3/5	0.86	6/7	86
2/9	0.57	4/7	57
2/7	0.17	1/6	17
2/5	0.89	8/9	89
2/3	0.13	1/8	13
1/9	0.63	5/8	63
1/8	0.44	4/9	44
1/7	0.38	3/8	38
1/5	0.88	7/8	88
1/4	0.71	5/7	71
1/3	0.29	2/7	29

Note: Each trial was presented in two forms (as all the non-symbolic magnitudes appeared as both continuous and discrete pictures).

with respect to continuous and discrete pictures. Each trial consisted of a picture (1500 ms), followed by a mask composed of scrambled black and white dots (150 ms), then number. On the number screen, participants were allowed up to 7000 ms to input their answer. They were told to select a key labelled "picture" (the "a" key) or a key labelled "number" (the "l") key for whichever proportion they thought was larger. The positions of these keys were reversed for half of the participants. No feedback about accuracy was given.

Results and discussion

Accuracy for each condition was averaged across participants. Mean accuracy was 86% for each of the three number types. A 3 × 2 within-subjects ANOVA yielded no reliable effects of number type, picture type or their interaction. Accordingly, our analyses focus solely on response times (RTs).

RTs for correct trials were averaged across participants (see Figure 3). A 3 × 2 within-subjects ANOVA yielded a reliable influence of picture type such that continuous pictures were evaluated significantly faster than discrete pictures (continuous: 1301 ms vs. discrete: 1444 ms; $F(1, 86) = 20.05$, MSE = 6.8, $p < 0.001$, $\eta_p^2 = 0.318$). There was also a significant effect of number type ($F(2, 86) = 30.58$, $p < 0.001$, $\eta_p^2 = 0.416$) such that trials with fractions were slower than trials with either decimals (fractions: 1481 ms vs. decimals: 1292 ms; $t(43) = 7.0$, $p < 0.001$) or percentages (fractions: 1481 ms vs. percentages: 1344 ms; $t(43) = 4.8$, $p < 0.001$). The interaction between number type and picture type was not reliable ($F(2, 86) = 0.47$, MSE = 2.5, $p = 0.63$, $\eta_p^2 = 0.011$).

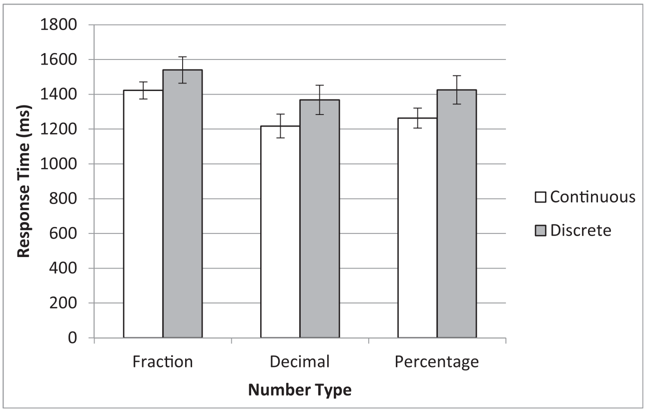

Figure 3. Mean correct response times for fractions, decimals and percentages for continuous versus discrete pictures (Experiment 1). Error bars indicate ± 1 standard error of the mean.

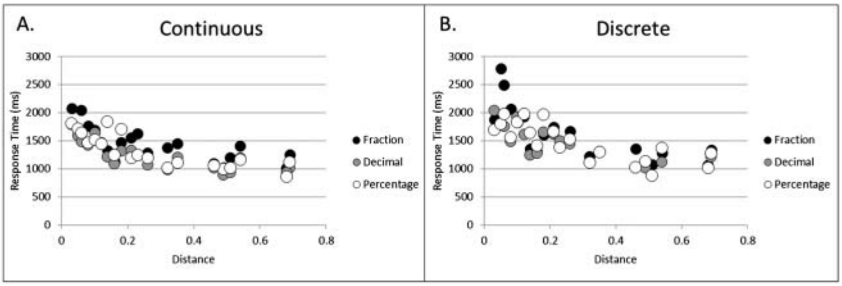

Figure 4. Mean correct response times for continuous (panel A) and discrete (panel B) trials based on the numerical distance between non-symbolic (picture) and symbolic (numerical) proportions, for each of the three number types (Experiment 1).

In order to examine the symbolic distance effect, we averaged RTs for correct trials across trials (see Figure 4). We then performed regressions on each of the trial types across number and picture types, using the logarithm of the distance between the non-symbolic and symbolic proportions as a predictor for RTs (see DeWolf et al., 2014). As summarised in Table 2, beta weights were reliable for all comparisons, indicating that a distance effect (reduced RTs as numerical distance increased) was observed for each combination of number type and picture type. In contrast to previous studies (DeWolf et al., 2014; DeWolf et al., 2016) in which fractions yielded a steeper slope than other number types, slopes did not differ reliably across number types in Experiment 1. The lack of a slope difference in the present study is likely a floor effect attributable to the relative simplicity of the stimuli used in the present experiment (i.e., fractions all involved single-digit numerators and denominators).

In summary, Experiment 1 demonstrated an advantage in processing speed for decimals over fractions in comparing the magnitude of a number with that of a non-symbolic picture. The percentage format produced essentially the same advantage as the decimal format. It is possible that people tend to convert fractions into a continuous code (e.g., by dividing) prior to making magnitude comparisons. The advantage of the metric formats held for discrete as well as continuous displays; however, comparisons were slower overall when the picture showed a discrete rather than continuous

Table 2. Results of regression analyses based on log numerical distance (Experiment 1).

Picture type	Number type	Beta	Significance
Discrete	Fraction	−0.96	$t(18) = 7.4, p < 0.001$
	Decimal	−0.72	$t(18) = 7.2, p < 0.001$
	Percentage	−0.67	$t(18) = 5.16, p < 0.001$
Continuous	Fraction	−0.63	$t(18) = 6.22, p < 0.001$
	Decimal	−0.59	$t(18) = 8.57, p < 0.001$
	Percentage	−0.65	$t(18) = 7.24, p < 0.001$

magnitude, suggesting that people may have converted the discrete picture into a continuous magnitude prior to comparing it with a number. This finding provides additional evidence that metric numerical magnitudes are linked more closely to continuous than to discrete quantities. Potential alternative interpretations will be considered in the General Discussion.

Experiment 2

Experiment 1 demonstrated an advantage of decimals and percentages relative to fractions in comparing magnitudes of rational numbers to those of pictures. Using closely matched stimuli, Experiment 2 examined the pattern of alignment in a task that required judgements about quantitative relations (alternative ratios between quantities). This laboratory test of relational processing is sufficiently challenging that it can be used with college students operating under speed pressure (DeWolf et al., 2015a; Lee et al., 2016; Plummer et al., 2017). The task has the critical property that it can be performed successfully without calculating the magnitude of any number. Similar relational tasks taught in school include judging equivalence of different fractions (DeWolf, Bassok, & Holyoak, 2015b, 2016) and multiplying by reciprocals (DeWolf, Son, Bassok, & Holyoak, 2017). The knowledge required to make relational judgements such as these is quite different in nature from the pure measure of magnitude understanding provided by the magnitude comparison task used in Experiment 1. Based on the analysis of rational numbers presented earlier, we hypothesised that the relation task would show an advantage for fractions paired with discrete displays.

Method

Participants

Participants were 60 UCLA undergraduate students (mean age = 20.7, 44 female) who received course credit for participating.

Materials and design

The study was a 3 (number type: fraction, decimal, percentage) × 2 (picture type: discrete vs. continuous) × 2 (relation type: part-to-part vs. part-to-whole) within-subjects design. A part-to-part ratio (PPR) is the relation between one subset and another subset that together make up a whole quantity (i.e., what is often termed a "ratio"). A part-to-whole ratio (PWR) is the relation between one subset and the whole (i.e., a "proportion"). Figure 5 illustrates an example stimulus. In this example, a discrete picture containing two grey dots and four black dots could represent a PWR: two grey dots out of six dots total (2/6). Alternatively, this same discrete display could represent a PPR: two grey dots compared to four black dots (2/4). This type of distinction between ratio types

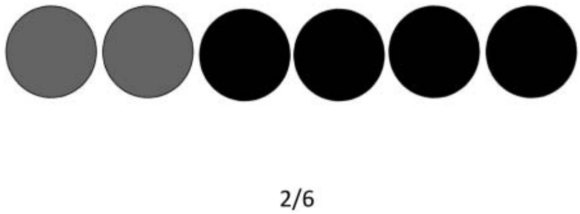

Figure 5. Example picture-plus-number stimulus (Experiment 2). The number specified in this trial is 2/6, making the correct response PWR. For the corresponding PPR problem, the number paired with this picture would be 2/4.

is one that creates difficulty for set-inclusion reasoning in young children (e.g., McCabe, Siegel, Spence, & Wilkinson, 1982). For example, a child shown a set of flowers that includes a number of roses and a lesser number of tulips will sometimes erroneously report that "there are more roses than flowers", apparently confusing the salient PPR with the part–whole relation.

Note that a display could represent either of two possible PWR relations (e.g., Figure 5 depicts both 2/6 and 4/6 as PWR relations). However, once the number is presented, it will match at most one of these two PWR possibilities, thereby disambiguating which case is relevant on the trial. It is possible that some inhibitory process is required to suppress the irrelevant PWR. This inhibition of the irrelevant ratio might slow processing of PWR relations relative to PPR relations (although it is also possible that both PWR relations would have to be suppressed on PPR trials). In any case, our hypotheses focus on the conceptual match between fractions and discrete displays, rather than on the relative difficulty of identifying PWR versus PPR relations.

The stimulus on each trial consisted of a picture that was either continuous or discrete, and a symbolic number represented as either a fraction, decimal or percentage. The images representing non-symbolic quantities were very similar to those used in Experiment 1, except that the colour of the smaller subset varied randomly from trial to trial (i.e., for half of the trials the smaller subset was grey, and in the other half of the trials the smaller subset was black). The value of the rational number displayed with a particular picture on each trial corresponded to either the PPR or PWR shown in the picture. The participant's task was to decide which of the two relationships (PPR or PWR) matched the number and picture combination. Since the colour of the smaller subset varied randomly, the colour that participants needed to consider to correctly identify a PWR relation varied from trial to trial, and could not be determined until the number was presented and considered in relation to the picture. Both accuracy and RTs were collected.

Procedure

Participants received the following instructions: "In this experiment, you will see a display followed by a value. The display and value will show one of two relationships." Below this statement, two different displays were presented, showing the PPR and PWR relations, respectively. In instructions to participants, the PPR relation was referred to simply as "R1", and the PWR relation was referred to as "R2". Each display was shown with the corresponding symbolic notation in fraction, decimal and percentage forms that matched the relevant ratio. Participants read the following description of each relation: "R1: value represents the ratio of one subset to the other subset (part-to-part). R2: value represents the ratio of one subset to the total set (part-to-whole)." Finally, participants were told that their task on each trial was to identify which of these two relationships corresponded to the number paired with the picture. Participants were instructed to press the "a" key to indicate R1 and the "l" key to indicate R2.

The experiment consisted of 120 trials, blocked by number type (40 trials per block). The order of blocks was counterbalanced across participants (using all six possible orders of the three blocks). Within each block, trial order and picture type (continuous versus discrete) were varied randomly. On each trial, a fixation cross was presented for 500 ms. Then, the cross disappeared and the picture appeared at the top of the screen. After 1500 ms, the number appeared below the picture, which remained visible. After the number appeared, participants were allowed up to 7000 ms to input their answer. The next trial began when the response was given or after the maximum time had elapsed. The trial structure was virtually identical to that used in Experiment 1, except that in Experiment 2, the picture remained visible until the participant responded (due to the greater overall difficulty of the relation judgement task).

Prior to beginning each block, participants completed four practice trials, one for each combination of factors (continuous and discrete pictures for both PPR and PWR relations, all paired with the number type used in that block). After inputting a response, participants were shown the correct response and an explanation for why that relation was correct. After completing the practice trials, participants were told that they would begin the test trials. At this point, they were instructed to respond as quickly as possible without sacrificing accuracy. In contrast to Experiment 1, feedback (the text "Response Incorrect") was shown after each trial that resulted in an error (due to the greater difficulty of the task in Experiment 2). The feedback message remained on the screen until participants pressed a key to advance to the next trial.

Results and discussion

Data from three participants were excluded from analysis: one who did not complete all blocks, and two who pressed the same key throughout most of

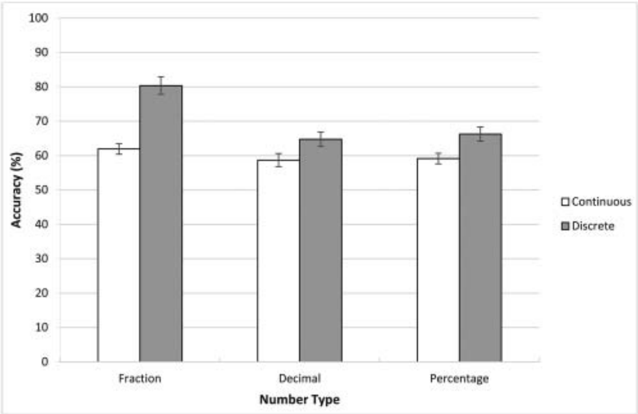

Figure 6. Mean accuracy for fractions, decimals and percentages for continuous versus discrete pictures (Experiment 2). Error bars indicate ± 1 standard error of the mean.

the experiment, with RTs consistently under 500 ms. Analyses were based on the remaining 57 participants. Accuracy and mean RTs on correct trials were calculated for each condition for each participant. An initial omnibus ANOVA found that the interaction between number type and display type did not vary across the two ratio types ($F(2, 112) = 1.532$, MSE = 0.018, $p = 0.221$, $\eta_p^2 = 0.027$). Accordingly, all analyses are reported after collapsing across the factor of ratio type.

Figure 6 displays the pattern of accuracy for each condition. Task difficulty was much greater for the relation identification task than for magnitude comparisons (Experiment 1); however, mean accuracy exceeded chance (50%) for all conditions. A 3 (number type: decimal, fraction, percentage) $\times$ 2 (display type: continuous, discrete) within-subjects ANOVA was used to assess differences in accuracy and also in RT. A significant interaction was obtained between display type and number type ($F(2, 112) = 11.212$, MSE = 0.012, $p < 0.001$, $\eta_p^2 = 0.167$). For discrete displays, participants were more accurate when using fractions (80%) than decimals (65%) or percentages (66%). Planned comparisons revealed that the difference between fractions and decimals was statistically reliable ($t(56) = 6.34$, $p < 0.001$), as was the difference between fractions and percentages ($t(56) = 5.46$, $p < 0.001$). Accuracy for decimals and percentages did not differ ($t(56) = 0.62$, $p = 0.54$). For continuous displays, accuracy did not differ across number types ($F(2, 112) = 1.47$, MSE = 0.012, $p = 0.24$, $\eta_p^2 = 0.025$).

Figure 7 shows mean RTs for correct trials for each condition. The interaction between display type and number type was reliable ($F(2, 112) = 13.57$, MSE = 146,653, $p < 0.001$, $\eta_p^2 = 0.195$). For discrete displays, mean RTs were faster for fractions (1925 ms) than for decimals (2200 ms) or percentages (2289 ms). The difference between fractions and decimals was reliable

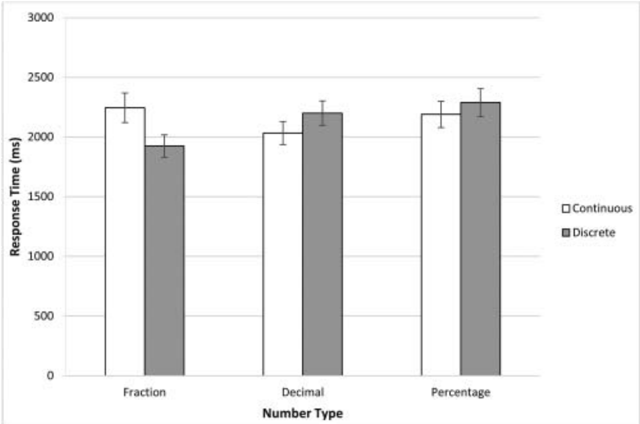

Figure 7. Mean correct response times for fractions, decimals and percentages for continuous versus discrete pictures (Experiment 2). Error bars indicate ± 1 standard error of the mean.

($t(56) = 3.05$, $p = 0.003$), as was the difference between fractions and percentages ($t(56) = 3.81$, $p < 0.001$). RTs for decimals and percentages did not differ significantly ($t(56) = 0.96$, $p = 0.34$). The RT analyses thus confirm that the accuracy advantage obtained for fractions over decimals and percentages is not attributable to speed-accuracy trade-offs.

Overall, the results of Experiment 2 revealed that in a task requiring relation identification, an advantage for fractions over both decimals and percentages was observed both in accuracy and RT, but only for discrete displays. The findings for fractions and decimals replicate those obtained by DeWolf et al. (2015a) (also Lee et al., 2016; Plummer et al., 2017). Accuracy was the highest and RTs the lowest for fractions paired with discrete pictures, indicating that the explicit relational structure of fractions conferred an advantage for these relational judgements. In discrete displays, the numerator and denominator of a fraction can be mapped directly to the countable subsets in the display. Metric formats for rational numbers, lacking this internal relational structure, cannot be mapped to countable subsets without manipulation. As a consequence, participants have to rely on estimation strategies, which are more error-prone than counting.

General discussion

Summary

Across two experiments using closely matched stimuli, a dissociation was observed between performance on magnitude comparisons versus relation judgements for different rational number formats. Specifically, we tested the hypotheses that decimals are conceptually linked to continuous mass quantities and naturally express magnitude, whereas fractions are conceptually linked to discrete, countable quantities and naturally express relations between subsets. The present results support these hypotheses. More broadly, these experiments support the hypothesis that rational numbers are subject to semantic alignment. Rather than simply serving as notational variants, different formats of numbers are naturally well-suited to represent different kinds of real-world quantities and relations between them

In Experiment 1, participants compared magnitudes of symbolic and non-symbolic quantities. For both continuous and discrete displays, RTs were faster for comparisons involving decimals or percentages than for comparisons involving fractions. RTs were faster overall for continuous displays than for discrete displays. In addition, we observed symbolic distance effects for each combination of number and display type, as RTs decreased with increases in the numerical distance between the magnitude shown in the picture and the magnitude of the number paired with it. These results suggest that metric formats (decimals and percentages) naturally lend themselves to magnitude comparison tasks, and are better aligned with continuous non-symbolic representations of magnitude. Discrete displays were processed more slowly than continuous displays, suggesting that discrete displays were converted into a continuous code in order to perform the magnitude comparison task.

In Experiment 2, participants made judgements about ratio relations based on a symbolic and non-symbolic quantity. For discrete displays, accuracy was the highest and RT the fastest for judgements involving fractions, whereas performance did not differ between decimals and percentages. This finding supports the hypothesis that the two-dimensional format of fractions is well-suited for expressing relations between countable sets. Decimals and percentages lack this internal relational structure, and hence tend to evoke estimation strategies (Plummer et al., 2017), which are more error-prone than counting. It is worth emphasizing that the benefit afforded by fractions and discrete pictures reflects the joint contributions of both – a discrete picture provides countable elements that allow for exact calculation, while the bipartite structure of a fraction provides an explicit relational structure to which those counted elements can be mapped.

The present study is the first to compare performance on these tasks for percentages as well as decimals and fractions. In both experiments, percentages behaved very much like decimals, yielding faster decisions than fractions for magnitude comparisons (Experiment 1), but less accurate and slower decisions than fractions for relational judgements with discrete quantities (Experiment 2). These findings indicate that although in some cases percentages may evoke the concept of proportion, as do fractions (Parker & Leinhardt, 1995), their dominant interpretation is similar to that of decimals.

Whereas fractions are inherently two dimensional in nature, percentages have an implicit fixed denominator (100), and therefore are one-dimensional-like decimals (Halford et al., 1998, 2010). The close linkage between percentages and decimals is consistent with work suggesting that it can be effective to introduce rational numbers first using percentages, followed by decimals and then fractions (Kalchman, Moss, & Case, 2001; Moss & Case, 1999).

Implications for processing of fractions

The differences in the effectiveness of different number types across the two tasks may reflect flexibility in the representations people use to process fractions. Evidence suggests that people can operate on fractions either as holistic magnitudes or in terms of their discrete components (numerator and denominator). Previous studies of magnitude comparisons with fractions indicate that adults sometimes process only the whole-number components of the fraction (e.g., Bonato, Fabbri, Umiltà, & Zorzi, 2007; Fazio, DeWolf, & Siegler, 2016). But when the stimulus set is constructed so that reliable judgements require consideration of the holistic magnitude (as in our Experiment 1), a clear distance effect is obtained with fractions (e.g., Schneider & Siegler, 2010).

Few previous studies (e.g., Matthews & Chesney, 2015) had examined how holistic processing of rational number magnitudes may vary depending on the symbolic or non-symbolic nature of the representation. The present Experiment 1 is the first to investigate magnitude comparisons when both the type of non-symbolic quantity (continuous vs. discrete) and the type of symbolic number format (fraction, decimal, whole number) are systematically varied. We found that for all number formats, comparisons were facilitated when the non-symbolic quantity was continuous rather than discrete. In addition, the task was easier when the numbers were decimals or percentages rather than fractions. The overall pattern of results supports the hypothesis that people prefer to make magnitude comparisons using an internal representation akin to a continuous number line. Metric symbolic formats (decimals and percentages) convey continuous magnitudes more directly than do fractions, but fractions also are converted into continuous magnitudes in order to compare them to non-symbolic quantities.

The way in which proportions are represented (both symbolically and non-symbolically) has implications for understanding and isolating relationships between entities. In contrast to the magnitude comparison task, the relation-identification task (Experiment 2) favours componential processing of fractions (mapping the numerator and denominator onto distinct components of the visual display). In Experiment 2, one might have expected fractions to lead to superior performance even when the display showed a continuous quantity, since it would seem possible to mentally impose discrete divisions that could then be counted. However, a recent study using eye-tracking methods found no evidence of a counting strategy being applied to continuous displays even when paired with a fraction (Plummer et al., 2017). It appears that "mental discretisation" is difficult for the type of continuous displays used in the present study.

It could be argued that the fraction advantage observed in Experiment 2 was due to the fact that the fraction elements were always in a one-to-one relationship with the discrete displays (e.g. a PWR relation involving three red segments out of seven total segments paired with the fraction 3/7). However, DeWolf et al. (2015a) demonstrated that a similar fraction advantage with discrete displays could also be obtained in a more complex analogy task, even when the fraction elements were *not* in one-to-one correspondence with the display (e.g., three out of seven items paired with 6/14). Thus, it appears that the fraction advantage is not limited to cases involving a one-to-one mapping between fraction components and display elements.

Implications for reasoning and learning

The fact that percentages behaved very much like decimals in both tasks examined in the present study is consistent with other work on reasoning. The fixed denominator (100) associated with percentages implies that they (unlike fractions) do not convey "natural frequencies" in a population. In a variety of inference tasks, an advantage has been found for natural frequency formats over percentages, probabilities and other formats for which the size of the specified population has been removed or standardised (Gigerenzer & Hoffrage, 1995; Hoffrage, Gigerenzer, Krauss, & Martignon, 2002; Tversky & Kahneman, 1983). The present study shows a similar advantage for fractions over percentages in a different type of reasoning task – relation identification. Nonetheless, there may well be other reasoning tasks in which the percentage format is particularly effective. For example, percentages seem to be preferred to decimals or fractions when conveying quantities construed as rates (e.g., "the annual interest rate is 5%") or expressions of uncertainty ("there's a 50% chance of rain today"). Future research will be needed to determine if indeed there is an advantage for percentages over decimals in certain reasoning tasks, or whether percentages remain in use simply as an accident of mathematical history.

The impact of the type of non-symbolic magnitude representation has important implications for both magnitude representation and relational reasoning. It appears that continuous entities are better suited for magnitude estimation, whereas discrete entities allow for easier isolation of the sub-components of a ratio. Both Rapp et al. (2015) and Lee et al. (2016) found that educators adhere to certain conventions when using different types of symbolic rational numbers to reference relationships or quantities that are either discrete or continuous. In particular, decimals are more often used with continuous entities whereas fractions are recruited more often for discrete entities. These findings, coupled with the results of the current study, suggest that for both symbolic and non-symbolic representations, different quantity types may be differentially effective in teaching magnitudes versus relational reasoning.

Further, the relational structure of fractions may play an important role in preparing students to learn more abstract mathematics. For example, understanding fractions appears to be crucial for grasping the concept of a reciprocal relation (DeWolf et al., 2017). In addition, a fraction exemplifies an important type of duality. A fraction is at once a relationship between two quantities, expressed as a/b, and also the magnitude corresponding to the division of a by b. Thus, fractions can either be interpreted as the result of a division operation (yielding a magnitude understanding) or as a specific comparison between two sets. Similar dualities arise in algebra (Sfard & Linchevski, 1994). For example, the expression $5x$ indicates a relation between the magnitude of 5 and that of some unknown number x. This relation is meaningful even though the magnitude of the overall expression cannot be known prior to solving for the variable. Recent work has shown that measures of relational understanding with fractions predict early success in learning algebra (DeWolf et al., 2015b, 2016). More generally, research on the differential affordances of alternative mathematical notations may provide a bridge between numerical cognition and higher order reasoning.

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