COGNITION AND INSTRUCTION, *22*(1), 37–60

Analogy Use in Eighth-Grade Mathematics Classrooms

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Analogical reasoning has long been believed to play a central role in mathematics learning and problem solving (see Genter, Holyoak, & Kokinov, 2001); however, little is known about how analogy is used in everyday instructional contexts. This article examines analogies produced in naturally occurring U.S. mathematics lessons to explore patterns between analogy generation and instructional context. One hundred and three analogies were identified in a random sample of 25 eighth-grade mathematics classrooms videotaped as part of the Third International Mathematics and Science Study (see Stigler et al., 1999). Qualitative codes were used to gather quantitative data about the set of analogies, and emergent patterns considered. Specifically, the study examines patterns of teacher–student participation, analogy source and target construction, and contexts for analogy production. These data suggest that teachers regularly use analogy as instructional mechanisms to teach concepts and procedures, differentially generating sources to match the content goal of the analogy. The source and target construction is also related to whether the analogy is in response to students' needs for help. Teachers typically maintain control of each analogy by producing the majority of the comparison, though students are regularly involved in the components of the analogy that require minimal analogical thinking.

Empirical researchers across disciplines have argued that analogical reasoning may be central to learning of abstract concepts (e.g., Brown & Kane, 1988; Gentner, Holyoak, & Kokinov, 2001), procedures (Goswami, 1992; Ross, 1987), novel mathematics (Bassok, 2001; Novick & Holyoak, 1991; Ross, 1987), and to the ability to transfer representations across contexts (Novick, 1988; Reed, Dempster, & Ettinger, 1985). Analogical reasoning, the ability to perceive and operate on the basis of corresponding structural similarity in objects whose surface

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features are not necessarily similar, is also deemed an essential part of the human capacity to adapt to novel contexts (Holyoak & Thagard, 1995) and to play a role in children's cognitive repertoire for learning about the world (for review, see Goswami, 2001), possibly since infancy (Chen, Sanchez, & Campbell, 1997). Furthermore, analogies have become typical components of standardized testing (e.g., Scholastic Assessment Test, Graduate Record Exam) because they are believed to involve basic reasoning skills.

Despite these indications that analogy may be an important component of human thinking and learning, currently, little is known about analogy as an instructional device in everyday practice. Traditional studies of analogical reasoning require participants to complete a formal higher order relationship such that "a" is related to "b" in the same way as "c" is related to "d," typically notated as *a: b:: c: d* (Goswami & Brown, 1989; Inhelder & Piaget, 1964; Sternberg, 1977). In an alternative format, prior information provides a potential source analogy that can be used to guide a participant reasoner's problem-solving strategy when asked to solve a target problem (Gick & Holyoak, 1980, 1983). For example, Gick and Holyoak (1980) provided participants with a story in which a general attacked a castle simultaneously from multiple directions. Then participants were given Duncker's classic radiation problem in which they are asked to determine how a doctor could use limited strength lasers to eliminate a patient's tumor. The correct solution requires the doctor to attack the tumor from multiple directions simultaneously—a strategy that follows the same structure as the castle story. Participants were very unlikely to generate the solution to Duncker's radiation problem spontaneously, but those made aware that they could use the prior problem to aid in generating a solution were highly successful at transferring the structure across problem contexts.

Research studies like these have delineated the cognitive steps that occur when people reason on the basis of analogical comparisons (e.g., Holyoak, Novick, & Metz, 1994) and have provided fundamental insights into how features of the source and target analogs affect reasoners' processing. For example, research has demonstrated that the level of *surface similarity* between source and target objects (salient feature overlap between objects) directly affects the difficulty level of noticing and using *relational similarity* (correspondences between objects that play parallel roles in the source and target; e.g., Gentner & Toupin, 1986; Holyoak, Junn, & Billman, 1984). In addition, there is evidence that analogical comparison can result in formation of abstract schemas to represent the underlying structure of source and target objects, thereby enhancing reasoners' capacity to transfer learning across contexts (e.g., Novick & Holyoak, 1991; Ross & Kennedy, 1990).

These findings suggest that analogical reasoning follows a series of specific steps that can lead to deep processing of novel information, yielding the capacity to learn and transfer knowledge to unfamiliar targets. However, typical experimental methodologies may not provide generalizable information about how analogy operates in more complex contexts or about how analogies are used in everyday interactions to teach and learn. Due to the structured nature of most studies, little information is available about the types of sources and targets selected by reasoners when the options are unconstrained. Further, few studies provide information about when people invoke analogy in everyday interactions (see Dunbar, 1995, 1997, 2001 for an exception).

In this project, we built on findings from the experimental literature on analogical reasoning to examine naturally occurring analogy in mathematics instruction. The mathematics classroom was selected as the site of the investigation because analogy has been implicated in many classroom activities including learning of abstract concepts and procedures, reasoning, and problem solving (e.g., Bassok, 2001; Goswami, 1992; Kolodner, 1997; Ross, 1987). Moreover, the abstract formal structure of mathematics makes analogical reasoning highly relevant to this domain. Mathematics is characterized by abstract structure in which underlying relationships remain the same but the object slots can be filled in varying ways. Noticing higher order similarity relationships between such instances of structural similarity is at the core of complex mathematical thinking (Bassok, 2001; Novick, 1988; Novick & Holyoak, 1991; Reed et al., 1985; Ross, 1987). For instance, students may not at first notice the similarity between addition of numbers and addition of variables because variables are different at the surface level. However with assistance or an increase in expertise, they could generate an analogy between these objects and use what they know about addition of numbers to inform their reasoning about addition involving variables. This simple case is illustrative of the power of analogy in enabling reasoners to apply information from one mathematical topic to other topics that are structurally similar.

In this study, we used qualitative codes to identify and gather quantitative information about teachers' discursive analogies to provide insight into teachers' typical methods of invoking and using analogy to communicate information. The codes were designed to gather information that cannot be captured in the laboratory but that map onto the types of questions regularly investigated in experimental studies in an effort to facilitate cross-fertilization of findings.

Accordingly, we identified the types of sources and targets selected by teachers and the level of similarity between these source and target objects. We also examined the instructional goals of the analogies, and the division of cognitive work between teachers and students.

Video data were analyzed from 25 U.S. eighth-grade mathematics classrooms videotaped as part of the Third International Mathematics and Science Study (TIMSS; Stigler, Gallimore, & Hiebert, 2000; Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999). Discourse analytic techniques were integrated with empirically derived models of analogical reasoning to explore analogy generation during teaching interactions. To generate data that could be compared across lessons, a definition of analogy based on Gentner's (1983) structure mapping model was used to identify units of analogy in all lessons. These units all contained an identifiable source object, a target object, and an explicit relational mapping constructed between these objects. With these criteria, all instances of analogy could be identified and compared across lessons in spite of surface differences in instructional content, providing a unique database of general analogy practices across U.S. mathematics classrooms. Subsequently, each unit was coded to gather generalized information about recurrent patterns across the set of analogies.

Our aim is twofold. First, we attempt to show that there are recurrent patterns in the way analogies are produced in this teaching context and to reveal the nature of these interrelated patterns. Second, we present ways to interpret teachers' naturally occurring actions using data from experimental literature. These findings make instructional analogies explicit and thus available for direct consideration by teachers, as well as providing means for applying learning data from experimenter-constructed studies of analogy to classroom practice.

METHOD

Data were collected as part of the TIMSS (complete information about TIMSS can be obtained at the Web site: http://nces.ed.gov/timss). Participants were 25 teachers and their students in eighth-grade mathematics classrooms across the United States. Specific teachers and specific lessons for videotaping were identified through a random probability sample of all lessons taught in public, private, and parochial schools in 1 year throughout the United States. Each selected classroom was videotaped on one occasion during a normal class period in a national video survey of mathematics teaching. The complete U.S. TIMSS video database contains 81 videotaped lessons (see Stigler et al., 1999).

Videotaped data from 25 lessons were randomly selected from the TIMSS sample for this study. Each lesson was taught by a different teacher, and the content areas were randomly selected from national eighth-grade curriculums (primarily algebra and geometry topics). Lessons were typically 50 min long yielding approximately 24 hr of videotaped data examined in this project.

Overview of Data Analysis

Lessons were analyzed using V-Prism 3.045 (Knoll & Stigler, 1999), a computer software system designed to allow simultaneous viewing of a digitized video and its typed transcript on a computer screen. In this software, the transcript moves temporally with the video and can be marked with research codes. Qualitative codes were used to generate quantitative information about patterns of analogy usage that relate to dimensions of analogical reasoning typically studied in experimental paradigms. Codes were derived from an iterative integration of empirical literature and data-driven observation, a "circle of analysis" to ensure that literature-driven codes were relevant to actual practice (Stigler et al., 2000).

Each lesson was analyzed in a series of seven passes. The data were divided between Lindsey E. Richland and a second coder for each pass, and approximately 20% were coded by both. Reliability between coders was calculated for the data coded by both. All disagreements were resolved by discussion. The first two passes resulted in identification of units of analogy. In five subsequent passes, these identified analogies were categorized on dimensions identified in experimental studies as relevant to the cognition operating during analogical reasoning. Specifically, each analogy was coded according to its source and target objects, the level of surface similarity between its source and target objects, its instructional function, and the participant organization of the steps involved in analogical reasoning. We describe the coding scheme and criteria for categorization for each pass in more detail next.

Pass 1 and 2: Identifying analogies. Completed occurrences of analogy were used as the unit of analysis for this study. All instances of verbalized analogies were first identified from within the 25 videotaped lessons. The definition of analogy used in this study was based on Gentner's (1983) structure mapping theory. Analogy was considered a comparative structure between known objects termed the *source* (or *base*) of the analogy and less known objects termed the *target* of the analogy. *Objects* are defined as entities that function as wholes at the given level of analysis. The source and target objects are aligned according to their *predicate* or relational structure such that inferences can be drawn from the source predicate structure to explain the target predicate structure.

Mapping is defined as the process of aligning and drawing inferences between the source and target objects. Mapping is the matching process that allows analogs from semantically distant domains to serve as relevant source and target objects. Objects and predicates from the source are aligned with objects and predicates in the target. Inferences are then drawn from the source structure and used to derive novel knowledge about the parallel target structure. We used a conservative measure of analogy, only marking units in which a source, a target, and an explicit structural mapping between the source and target could be identified in discourse or explicit gestures. These three components were considered essential to the occurrence of an analogy.

To maximize reliability in analogy identification, two passes were used. In a preliminary stage, we closely examined 5 lessons and identified all analogies produced by teachers or students. Recurrent keywords were selected from these analogies such that at least one keyword was present in every analogy. These included such words as *same, just like,* and *similar.* These words were then highlighted in the transcripts of the 25 sample lessons to generate a list of potential analogy units.

One coder additionally watched each lesson to identify any potential analogies not caught by the keywords.

Two coders then categorized every potential analogy unit as one of four codes. Units in which an explicit relational mapping was made between an identifiable source and an identifiable target were coded *relational mapping.* Units that were a continuation or a repetition of a previously identified relational mapping were coded *continuation.* Situations in which a relational mapping was suggested, but the source, target, and mapping were not all clearly present were coded *hints toward mapping.* Finally, marked instances that were not analogies were coded *not analogical mappings.* Sample analogy units identified are listed in Table 1. These are transcripts taken directly from the lesson discourse and represent the type of data used in all the coding. Reliability between the two coders was calculated on

TABLE 1
Sample Analogies Taken Verbatim From Classroom Transcripts

Transcript	*Verbatim Analogies Selected From Classroom Data*
1	Teacher: Now here's how I always looked at it. We're gonna say this—this circle right here is an orange. It's an orange. Alright, it's an orange. Now let's say we're gonna take—stick a needle in the orange n' suck out everything inside except for the peeling of the orange ((demonstrates with hands)). Okay (.) we're—we're gonna pretend like that's our circumference righ' there ((Teacher uses a pointer to run along the outside of the circle on the overhead projector))
2	Teacher: square root of five squared, I'm going to tell you right now, equals five. (Teacher aligns equations on board): $\sqrt{5^2} = 5$ $\sqrt{6^2} = ?$ Square root of six squared is what?
	Student: I know, six.
	Teacher: Six. Okay.
3	Teacher: Okay—just like equations, whatever you do to one side you have to do to the other ((gestures to left then right along with her words)).
	Students: ((overlap with the teacher)) to the other.
	Teacher: Whatever you do to the denominator ((curls hands inwards)), you have to do to the numerator ((uncurls hands to bend fingers outwards)).
	Students: ((overlap with the teacher)) numerator.
	Teacher: Okay (.) Now.
5	Teacher: A Rhombus … A rhombus is a square that's pushed over, correct ((teacher uses hands to gesture a square and then a rhombus))?
6	Teacher: Okay. The last thing we did last week was the circumference of the circle. The circumference is the outside of the circle. Today what I want to do is to go on the inside ((gestures with hand around the inside of a circle drawn on the board)) and cover the area of the interior of the circle. Alright and we did this almost the same way with polygons. We did the perimeter on the outside. Then on the inside we called it area. The inside of the circle is called area.

four lessons (18% of the total sample) and 105 potential analogy protocols. Agreement was 86%, or Cohen's κ = .65. Differences were resolved in discussion and consensus between the two coders.

Pass 3: Source–target construction. The most basic components of analogy are the source and target objects. The source and target objects are typically carefully constructed by an experimenter during laboratory studies of analogy, so little is currently known about how people naturally select sources and targets when constructing analogies. To gather general data about the categories of objects invoked by participants in the mathematics classrooms, the sources and targets of each analogy were classified as one of four types. The types were based on preliminary qualitative analyses of the data as well as on the categories of objects frequently used in laboratory studies.

Each source and target was coded either as

1. An outside-math phenomena (any nonmathematical object).
2. A schema (a general rule with no numbers or nonmath context).
3. A decontextualized math problem (a mathematics problem involving only numbers).
4. A contextualized math problem (a word problem set in a nonmath context).

In several cases, a teacher compared multiple sources to a single target or a source to a target stated multiple ways in which case the source or target was given a code of 5, multiple. See Table 2 for sample analogies of each structure.

Intercoder reliability was calculated separately for coding sources and targets. Twenty-two analogy protocols were used from three lessons (21% of the total analogies coded). Reliability was calculated to control for chance using Cohen's kappa, finding κ = .72 for sources and κ = .81 for targets.

Pass 4: Surface similarity. Analogies are constructions of similarity between the relational structures of the source and target objects. In contrast, the surface features of source and target objects should not be overly similar, at least according to a strict definition of analogy (Gentner, 1983). The level of surface similarity between source and target objects has been shown in laboratory studies to directly affect the difficulty that participants have in noticing and using relational correspondences between the relevant objects (Gick & Holyoak, 1980, 1983). The more featurally dissimilar, the more difficult it is for participants to notice relational similarity.

In this study, the surface similarity between source and target objects was coded to analyze the distance of inference teachers typically constructed within analogies. Furthermore, relationships could then be examined between the level of surface similarity produced and the context of the analogy. We were interested in

TABLE 2
Sample Analogies of Each Coded Level of Surface Similarity

Similarity	*Source*	*Target*
Far distance	Outside-math phenomena: "It's like balancing a scale, matter doesn't disappear, so to keep it balanced, whatever we do to undo one side we have to do to the other."	Decontextualized math problem: "It is divided by negative sixty, so we multiply by negative sixty on both sides."
Low similarity	Decontextualized math problem: "Ok, don't put all that other stuff. What if it was just 16/20. How would you reduce it?"	Decontextualized math problem: "Now if we do its with variables,. how would we reduce this?" $15xy^2z^4/25x^3y$
High similarity	Decontextualized math problem: "How would you multiply these?" $(x + 2y)\ (x = y)$	Decontextualized math problem: "In that case, how would you multiply $(5x + y)\ (x = 3y)$?"
Schema involved	Contextualized math problem: "Lets say you've got money. If you lost 88 cents and then you lost 5 cents more, whould you add or subtract to find out the total amount you lost?"	Math schema: "When you have a negative number minus another number, do you add or subtract?"

whether teachers invoked differing levels of similarity depending on their contextual goal for the analogy (Holyoak, 1985).

The surface-similarity code was closely related to the previous structure code. If the source was determined to be a code of 1, outside the math domain, and the target was any of the mathematical objects, the similarity was classified as *far distance.* If either the source or the target was determined to be a 2, schema or abstract rule for a mathematical principle, the similarity was classified as *schema involved.* Because schemas by definition have no surface features, their surface similarity to another object cannot be calculated in a meaningful way.

All other units were coded high or low surface similarity. *High similarity* was coded when the surface features of the source and target were similar such that the following were true: (a) the same number of solution steps were required to solve the source and target; (b) if there was a change in numbers, there was not a change in complexity of the type of number (e.g., integers to fractions); (c) if there was a change in variable, a different letter variable was used for the same slot in an equation; (d) geometric shapes were the same shape but different size; and (e) if the source and target problems were set in a context, the contexts were the same. *Low similarity* was coded when (a) there was a change in the number of steps necessary to solve a problem; (b) there was a change in the type of numbers used in the source and target; (c) a variable exchanged positions in a problem; (d) geometric objects revealed different shapes; or (e) if the source or target problems were set in a context, the contexts were different.

See Table 2 for examples of each level of similarity. Intercoder reliability was calculated on 17 analogy protocols from six lessons (16% of the total sample), $\kappa = .88$.

Pass 5: Instructional goal. Laboratory studies have been unable to provide information about the natural situations in which reasoners regularly produce analogies. In mathematics classrooms, this gap is particularly important to fill because teachers are continuously accomplishing many types of instructional tasks. We were interested in determining whether teachers would generate analogies more frequently to accomplish some types of instructional goals than others. Further, we sought to examine whether the instructional goal for producing an analogy was related to the types of sources and targets that teachers produced. Because experimental studies have examined learning implications for different source–target constructions, their results may provide insight into the learning by students that could be expected based on teachers' use of analogies.

A code was designed to capture the basic instructional goal underlying the teacher's use of analogy. Due to a wide variety of mathematical content and specific classroom dynamics, general codes were designed. These codes derived from observations of the videotaped lessons and standard descriptions of mathematical instruction.

Four types of instructional goals were coded, as illustrated in Table 3. The first code was *being a math student.* In several of the classrooms, teachers used analogies to give students a direction that was not content specific but rather was related to students' performance or work habits. These included instructing students to heighten their level of attention to a task, their focus on homework, or their performance on a test. The second code, *concepts only,* referred to analogies that were di-

TABLE 3
Sample Analogies Directed Towards Four Types of Mathematics Content

Being a Math Student	*Concepts Only*	*Concepts and Procedures*	*Procedures Only*
"Remove the parentheses, very carefully. Kind of like if you were a bomb squad called in to diffuse a bomb. If you mess up the first step, the whole problem will blow up in your face at the end!"	"When you're adding fractions, think about your denominators as units, like centimeters or feet. When you add length, the units must be the same in order to add them."	"Do you remember how we found the perimeter and area of polygons last week? This time it is the same concept but we are going to use similar formulas to solve for circumference and area of a circle."	"What were the steps we used to solve the last example? Now lets do the same thing in this problem. First lets factor the numerator and denominator and then we'll see what we can cancel."

rected toward explaining mathematics concepts. These were defined as mathematical phenomena that described the phenomenon by name, involved no reference to operations on specific numbers, and gave general rules such as "*z* is always bigger than *y*." Analogies that described the mathematical phenomena by name or gave general rules and gave specific procedural information about a problem were defined as *concepts and procedures.* In cases in which analogies were used to explain one or more problems, to aid with operations on a particular set of numbers, and did not give more general rules or name the mathematical concept, the code of *procedures only* was given.

Intercoder reliability was assessed on 23 protocols from three lessons (22% of total analogies coded), κ = .75.

Pass 6: Student context. To gain more insight into how teachers were using analogies, a code was developed to examine whether teachers were using the analogies in situations in which students were demonstrating difficulty. A code was used to assess whether analogies were being used specifically in contexts in which one or more students demonstrated lack of proficiency. This code also gives some insight into whether the analogies were part of a teacher's predesigned lecture or whether they were produced spontaneously in response to students' difficulty with class material. The discourse immediately preceding each analogy unit was examined to determine whether the analogy followed explicit evidence of students' mathematical trouble or whether it did not. Evidence included situations in which a student asked a question about the material, when a teacher asked a question that was not answered correctly by any member of the class, or when a teacher began an explanation with reference to a students' question or incorrect answer that was not caught by the videotape.

Intercoder reliability was calculated on 23 protocols from eight lessons (22% of total analogies), κ = .75.

Pass 7: Participant structure. Many of the analogies produced in the classrooms were collaboratively produced between teachers and students, and we were interested in which conceptual roles each type of participant performed. Analogies by definition are designed such that all recipients are at least somewhat familiar with the source object, enabling their participation in developing inferences about the target object and subsequent calculations. We were interested in whether teachers were utilizing analogies in this interactive way or whether they were completing the analogies and presenting students with completed comparisons.

The participant structure of each analogy was coded to determine whether it was generated by a teacher, student(s), or some combination of the two. Each analogy was deconstructed into segments using a modification of a componential analysis of analogical reasoning (Holyoak et al., 1994). A coding scheme was developed to identify the occurrence of five separate components: (a) representation of a source, (b) representation of a target, (c) finding a mapping, (d) making inferences and adapting them to the target, and (e) providing a solution to the target problem. When any of these components were not explicitly verbalized or visually represented, it was coded as *not explicit.* Each component was identified and the person who produced the component was recorded. These were coded *teacher* if the teacher alone produced the evidence of the component, *student* if one student alone produced the evidence of the component, and *class* if the component was accomplished by combined effort of more than one person, either multiple students or the teacher and student(s).

These components may be demonstrated using an analogy recorded previously in Table 1. In this analogy, the source is developed by the teacher in the first eight lines:

1 Teacher: Now here's how I always looked at it
2 We're gonna say this—this circle
3 right here is an orange. It's an orange.
4 Alright, it's an orange.
5 Now let's say we're gonna take—stick a needle in the
6 orange n' suck out everything inside except for the
7 peeling of the orange
8 ((demonstrates this process using hand gestures)).

The teacher is building a representation of a circle that highlights the relation between an orange peel and the outside of an orange. Next, the teacher makes an explicit link mapping his representation of an orange peel onto the circumference of a circle in lines 9 and 10:

9 Okay we're go—we're gonna pretend like that's our
10 circumference right there.

He adapts this mapping to demonstrate the inference that circumference is the outside rim of a circle just like the peel is the outside of an orange using a gesture recorded in lines 11 and 12:

11 ((Teacher uses a pointer to run along the outside
12 edge of the circle on the overhead projector)).

The final component analyzed, solving, was not relevant in this case and so was recorded as not explicit.

Coders also marked specific information about the level of elaboration of the source and of the mapping between source and target phenomena. Observation of the data suggested that analogy sources were sometimes discussed and elaborated more than was necessary to ensure that an analogy recipient would recognize the source object. Thus, if there was elaboration beyond the point necessary for a source to be recognized (e.g., if the teacher just mentioned had said "imagine an orange peel"), the participant producing the elaboration was coded. The level of elaboration is interesting because elaborations may organize the listener's interpretation of the target phenomenon and assist in generating certain types of mappings and inferences and also help the person generating an analogy to engage the listener as a participant. The process of engaging novices as active participants has been believed to be an important component of teaching according to constructivist models of learning. In the context of the classroom, the elaboration could allow a teacher to maintain control over the information being explained in an analogy while engaging students through elaborative questions.

The mappings were marked either "implicit" or "explicit" to describe the level of explication used to confirm the relationship between the source and target items for a recipient of the analogy. Experimental data have suggested that more explicit information about the links between source and target items can assist participants in mapping analogically (e.g., Novick & Holyoak, 1991), so this factor might be an important predictor of students' learning from presented analogies.

The participation, elaboration, and mapping level of explanation were all included in a single coding system. Reliability was calculated between two coders on 77 analogy component protocols—derived from 13 analogies from nine lessons (13% of the total sample), $\kappa = .69$.

RESULTS

Production of Analogies

A total of 103 analogies were identified. Each lesson had a mean of 4.1 analogies ($SD = 2.6$). The range for each lesson was between 1 and 11 analogies. Thus, in every lesson examined, at least 1 explicit analogy unit was coded. In most lessons, teachers also produced hints toward mappings indicating that students may have been performing mappings at teachers' prompts. In the sample of 25 lessons, 109 total hints were produced, with a mean of 4.5 per lesson and a range of 1 to 11.

Structure

Table 4 shows the frequency distribution of the analogy sources and targets in five categories: abstract math schemas, numerical math problems, quantitative problems set in a nonmath context, phenomena outside the domain of mathematics, and multiple types. Table 4 also shows the frequency with which each category of

TABLE 4
Frequency Distribution of Source and Target Structure

	Target Structure					
Source Structure	*Nonmath*	*Contextualized*	*Decontextualized*	*Schema*	*Multiple*	*Total*
Nonmath	1	2	4	8	0	15
Contextualized	0	12	4	3	0	19
Decontextualized	0	1	32	7	1	41
Schema	0	1	1	8	2	12
Multiple	0	2	4	8	2	16
Totals	1	18	45	34	5	103

source was paired with categories of targets. Both the sources and targets displayed significant differences between the frequency with which each structure category was used, $\chi^2(4, N = 103) = 26.5, p < .001$ for sources and $\chi^2(4, N = 103) = 68.4, p < .001$ for targets. Although both sources and targets were most likely to be decontextualized math problems (40% and 44%, respectively), all four categories were represented in both types of analogs.

There were discrepancies between types of objects generated as sources or targets. The most substantial distinction occurred between objects outside the domain of mathematics used as sources and targets, as 15% of sources were outside the domain of mathematics, whereas only 1 out of 103 targets was a nonmath phenomenon. This difference is not surprising because the math classroom is oriented toward mathematics learning.

The second largest proportion of targets were math schemas (33%), suggesting that teachers were using analogies not only to prompt solutions to single problems but also to aid in developing more general schemas. There were also many more instances of mathematical schemas used as targets than as sources. Only 12% of scores were schemas.

Finally, there were more multiple sources used in 16% of all analogies. Experimental work suggests that multiple sources are useful for schema acquisition (e.g., Gick & Holyoak, 1983), and this finding suggests that teachers may be implicitly aware of this fact.

Surface similarity. Table 5 shows the frequencies of the surface similarity between analogy sources and targets. There was no significant frequency difference among the four categories, $\chi^2(3, N = 103) = 3.058, p = .3$; all four types of analogy constructions were regularly used. Overall, 76% of analogies identified had minimal perceptual similarity between source and target objects. This finding suggests that in the classroom setting, teachers are more successful than typical laboratory participants at developing structural analogies.

TABLE 5
Frequency Distribution of Surface Similarity Between Source and Target Objects

Surface Similarity	*Frequency*
Schema involved	28
Far distance, outside-math source/target involved	19
Low surface similarity	31
High surface similarity	25
Total	103

Mathematical context. The total frequencies for the number of analogies directed toward specific mathematical content goals are shown in Table 6. The data indicate that some content goals are addressed by analogies more frequently than others, $\chi^2(3, N = 103) = 20.1$, $p < .001$.

Specifically, teachers were most likely to use analogies to teach procedures only (41%). The second most frequent function of analogies was to teach concepts and procedures in conjunction (29%); and 19% of the total number of analogies were directed to teaching concepts only. This result is particularly noteworthy because a relatively small portion of typical U.S. lessons is spent teaching concepts (Stigler et al., 1999). U.S. teachers spend more time teaching procedures than concepts during a single lesson, which suggests it is important that almost half of all analogies produced (46%) were directed toward teaching concepts either alone or with procedures.

Finally, analogies were also used for socialization purposes in 6% of cases. Teachers used analogies to describe optimal methods for being a math student. Analogy paradigms in laboratory studies are never designed with an emphasis on enacting change in human behavior other than developing a problem solution, so this is an interesting difference between the two settings.

Structure–context relationship. A further step was necessary to determine how teacher goals were related to the analogy that was generated. We first exam-

TABLE 6
Frequency Distribution of Mathematical Content in Analogies

Function	*Frequency*
Procedures only	42
Concepts and procedures	30
Concepts only	19
Being a math student	12
Total	103

ined whether the mathematical goal of the analogy was systematically related to the type of sources selected. Codes from the instructional goal variable were collapsed to examine the differences between analogies teaching mathematical procedures and analogies directed toward mathematical concepts. Concepts are discussed much less frequently overall than procedures in U.S. mathematics lessons (Stigler et. al., 1999), although half of the analogies identified in this data set were teaching concepts. All analogies in which concepts were discussed were collapsed into a single category, "concepts present." These included analogies initially coded *concepts only* and analogies coded *concepts and procedures*. Analogies coded *procedures only* remained the same. Because this analysis was focused on the type of mathematics being instructed, the more sociological analogies directed toward being a math student were excluded.

A Pearson chi-square test was performed to test the relationship between the preceding source structure variable with the mathematics instructional goal. This test revealed that there was a significant relationship between the source generated and the structure of each analogy, $\chi^2(4, N = 91) = 17.4, p < .001$. As shown in Figure 1, teachers systematically altered the categories of sources they selected depending on their mathematical goal for each analogy. When teaching concepts, teachers drew equally from the four types of sources, whereas for teaching procedures, there was a different pattern. For instance, teachers attempting to teach procedures were most likely to use noncontextualized mathematics problems as sources. Conversely, teachers almost never used non-math or schema sources to teach procedures only.

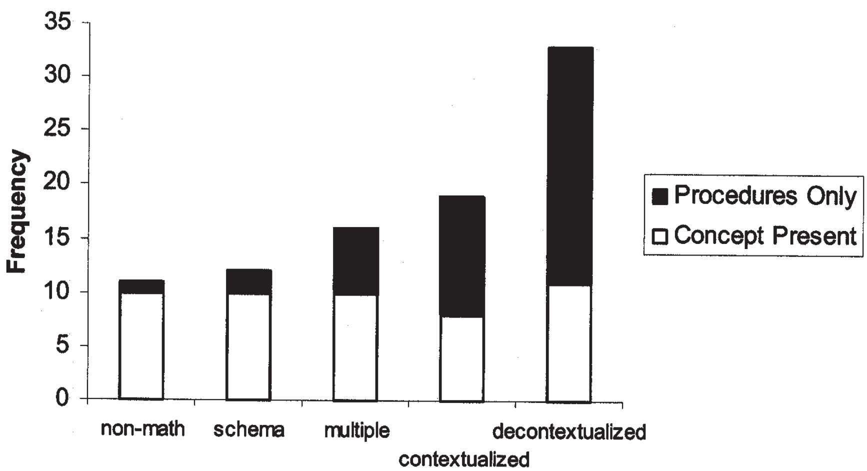

FIGURE 1 The relation between the teaching goal of an analogy and the type of source generated as the base of the analogy.

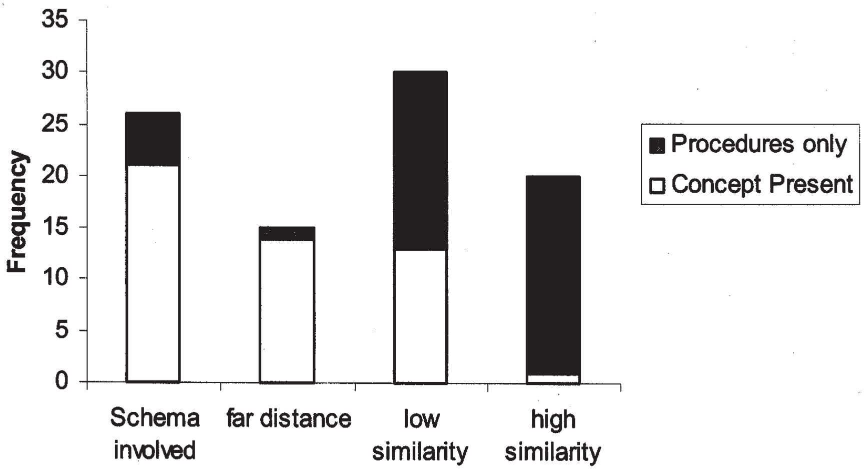

FIGURE 2 The relationship between analogy teaching goal and the level of surface similarity developed between source and target objects.

Second, to determine whether the overall similarity structure of analogies is related to their function, a Pearson chi-square test was performed to compare the surface similarity between the source and target of each analogy with the mathematical instructional goal of the analogy. This test revealed that there was a significant relationship between these variables, $\chi^2(3, N = 103) = 37.5, p < .0001$. As shown in Figure 2, several patterns are apparent. Far distance analogies were almost exclusively used to explain concepts or concepts paired with procedures. Schemas were also much more likely to be used in analogies demonstrating concepts than in analogies to teach procedures. In contrast, relational mappings with high surface similarity between sources and targets were almost exclusively used to teach procedures only.

Student context. There was no significant difference between the number of analogies produced following or not following a student's demonstration of difficulty with math material, $\chi^2(1, N = 103) = 1.17, p = .28$. Forty-six analogies were produced immediately after a student showed difficulty with a problem or concept, and 57 were not preceded by any demonstration of student difficulty. However, the analogies produced in each case had some different characteristics. First, a chi-square test was used to determine whether teachers used analogies for different mathematical functions in response to students having difficulty. The function variable was collapsed into the two categories examined previously, concept present or procedures only. As evident in Figure 3, the statistic revealed that teachers were significantly more likely to discuss concepts in analogies not following stu-

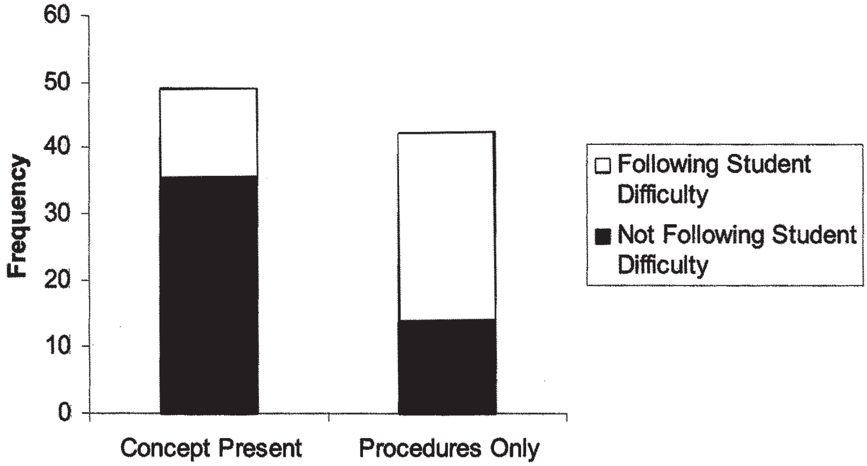

FIGURE 3 The relationship between the type of mathematical content taught in teachers' analogies following or not following students' explicit demonstration of difficulty with class material.

dent trouble, $\chi^2(1, N = 103) = 14.71, p < .001$, and in contrast were more likely to explain procedures only following student trouble.

Second, the surface similarity variable was examined in relation to the student trouble variable to determine whether teachers were manipulating the level of surface similarity between source and target objects according to the knowledge state of the student recipient. A chi-square analysis revealed that the level of similarity between source and target was significantly different across the two levels of the student trouble variable, $\chi^2(3, N = 103) = 18.20, p < .001$. The pattern apparent in Figure 4 indicates that teachers were much more likely to use far distance analogies and to use schemas when there were no signs of student trouble, whereas they were much more likely to use comparisons with high surface similarity to help students having difficulty. Empirical research has shown that surface similarity cues facilitate transfer (e.g., Holyoak & Koh, 1987), suggesting that teachers are using surface similarity in adaptive patterns to help students with immediate difficulties.

Participant structure. The frequencies of teacher and student participation are shown in Table 7. These data reveal that the analogies produced in classrooms were largely generated and constructed by teachers. Teachers generated 84% of all sources independently. In addition, in contrast to most experimental paradigms in which sources are presented to students in a single form, 81% of all sources were elaborated beyond what was necessary for the source item to be understood or referenced.

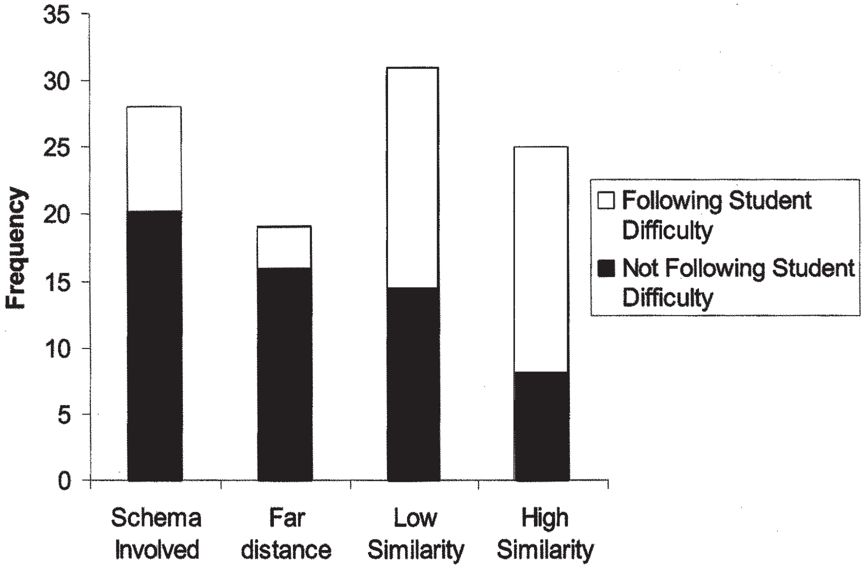

FIGURE 4 The level of surface similarity developed between source and target objects as a function of whether or not the analogy was generated in response to a student demonstrating difficulty with class material.

Teachers also produced 77% of the targets, suggesting that in the majority of these analogies, the teachers defined the problems or mathematics that were being explained. Even though almost half of these analogies were stimulated by student evidence of difficulty or by student questions about the material as measured previously, teachers were not generating analogies directly based on student formulations of the problem. This approach potentially eliminates an important learning opportunity for students.

TABLE 7
Frequency Distribution of Participant Roles During Analogy Production

	Component					
Participant	*Source Initiation*	*Source Elaboration*	*Target*	*Mapping*	*Inference and Adaptation*	*Problem Solution*
Teacher	86	39	79	92	44	4
Class	9	44	8	7	27	32
Student	8	0	16	4	1	7
Not explicit	0	20	0	0	31	60

Teachers also produced nearly 90% of all mapping statements, suggesting that teachers are much more highly involved in producing classroom analogies than are students. Students were regularly involved in assisting with elaborating source objects, making and adapting inferences, and in performing calculations to find final problem solutions. These data raise questions about the level of student participation in the analogical reasoning used by teachers when using analogy to facilitate mathematical understanding.

If the teacher is producing a large number of the analogies, it is unknown whether students are understanding the structural mapping underlying the analogy or are simply waiting for the teachers' interpretation. Students were fairly frequently involved in the inference and adaptation step in collaboration with the teacher or as multiple students (26%), but it is unclear how substantially students need to understand the analogies to participate in verbalizing this component.

To examine the distribution of participant roles within single analogies, a composite code was developed to assess the number of analogies in which teachers, students, or a combination produced the source, target, and mapping components. These data showed that in 66% of the 103 total analogies, teachers produced these three essential components. A student produced the source, target, and mapping in only 2% (2 analogies) of the total analogies, and in 33% of the units there was a combination of participants. This pattern yielded a significant difference in the participant roles in the production of communicated analogies, $\chi^2(2, N = 103) = 63.5$, $p < .001$, and indicates that teachers retain much more responsibility for production of mathematical analogies than they give students.

A final code assessed the explicitness of the mapping statements. The mapping between the source and target objects was made explicit in 72% of all analogies, whereas in 28% of cases, they were aligned either visually or by temporal sequencing. This difference was reliable, $\chi^2(1, N = 103) = 19.7$, $p < .001$, suggesting that communicative analogies are most likely to make the relationship between source and target items explicit. Such explicitness has been found to greatly assist participants in experimental paradigms to notice structural similarity and then to generate inferences about a target based on the source (e.g., Gick & Holyoak, 1983). Students, therefore, may be more likely to understand analogies in which there is an explicit mapping statement made, but this is not certain.

DISCUSSION

Several findings emerged from the data set that provide insight into the organization and production of analogies in naturally occurring mathematics classrooms. We discuss the data to explore the features of analogy usage relevant to both theoretical models of analogy and classroom teaching practices.

It is first evident from the data that verbal analogies are regularly produced in U.S. mathematics classrooms. These relational comparisons can be separated from pure similarity comparisons in which both surface and structural features would be similar (Gentner, 1983). In the classroom, almost one fourth of the analogies were far distance analogies, which suggests that analogy usage here may be quite different from typical laboratory situations in which participants have difficulty noticing and using analogies without explicit cues. The teachers consistently generated at least one, and typically multiple, analogies during the videotaped classroom period. Students were less likely to generate verbal analogies, possibly due to the asymmetric teacher/expert–student/novice relationship in the classroom.

Second, reliable relations were obtained between the instructional goal of the analogies and the type of analogies produced. Two main types of contextual goals were investigated: the mathematical content goal of the teaching interaction (mathematical context) and the students' states as "demonstrating trouble" or "not demonstrating trouble" (student context). Both of these contextual states were related to the type of source analogs generated by teachers.

The mathematical context was related to the source analogs generated in several specific patterns. First, nonmath sources were used almost exclusively when instructors were developing analogies to teach mathematical concepts and when using analogies for socialization purposes. Conversely, when explaining mathematical procedures, teachers were most likely to use noncontextualized math problems.

These data have several implications. First, these teachers' use of different types of sources depending on the goal of their analogy suggests that people may use surface similarity and source selection in more complex patterns than had been previously noted in experimental studies. In most experimental studies in which participants are given freedom to find a relevant source object, participants tend to make comparisons based on surface similarity if this is an option (e.g., Ross, 1987, 1989); however, teachers in this analysis opted to make distant analogies in specific goal-oriented situations. Thus, it is possible that participants' attention to surface similarity is related to their goals during the experimental tasks.

Second, teachers' use of distant analogies to teach concepts is worthy of notice, as it raises empirical questions about the nature of children's learning from analogies. Distant analogies have the capacity to allow novice students to reason about the novel concept by considering the source object, but it is unclear whether this occurs for students because teachers tend to perform all of the verbal work in constructing the relations. Distant analogies are often difficult for laboratory participants to notice and store, so it is important to know whether students are similarly having difficulty with these comparisons in the mathematics classroom.

The role of students' demonstrations of proficiency before the initiation of analogies is an additional area in which teachers varied the type of analogy they generated as a function of the instructional context. Specifically, there were reliable relations between whether or not the analogy followed student indications of difficulty with the structure and the type of mathematical content explored in the analogy. Teachers were most likely to generate analogies directed toward teaching procedures when the analogy followed student difficulty and concepts when the analogy did not follow student difficulty.

A related analysis of the surface similarity between sources and targets used to develop these analogies indicated that teachers were manipulating the level of similarity between source and target analogs depending on the immediate student context. Thus, when an analogy followed students' demonstration of trouble, the analogy was more likely to have high surface similarity between the source and target than when the analogy did not follow students' explicit demonstration of trouble. In addition, teachers were most likely to use far distance analogies and analogies involving schemas when the analogy did not follow student difficulty.

These findings suggest that teachers are explicitly or implicitly tailoring their analogy production to the cognitive needs of students. When students are showing difficulty, teachers generate sources and targets with higher surface similarity in a manner that reflects what decades of experimental research have determined to be reliable ways of making analogies more transparent for learners. Teachers seem to act according to an implicit assumption that students having difficulty will benefit most from highly transparent mappings between similar source and target analogs. Empirical studies have repeatedly shown that high surface similarity enhances participants' likelihood of noticing analogies and of transferring between problem contexts (e.g., Gick & Holyoak, 1980, 1983). However, it is possible that teachers' use of structurally similar procedure-based analogies facilitate students' immediate problem solving without providing the benefits of long-term transferable learning.

Furthermore, teachers tend to use more procedural than conceptual analogies when assisting students having trouble. This finding supports the position that teachers in these situations may be facilitating students' problem solving without providing them with information for longer term conceptual learning.

Close analysis of the participant structure for each analogy provides further insights into the reasoning opportunities available for students during verbalized analogies. Overall, it is evident that teachers typically generate the majority of the components of analogies produced, although there is effort made by teachers to engage students in collaborating in producing the comparisons. As a group, teachers independently generated the majority of the source, target, and mapping components of all analogies. The source and target are essential to generating an analogy, and finding a mapping is often described as the heart of analogical reasoning (e.g., Holyoak & Thagard, 1995).

At the same time, teachers seemed to operate from a constructivist perspective, desiring that students be active participants in the analogies. Although teachers may have been using analogies as explanatory tools, they made consistent efforts to engage student participation in elaborating the source, making and adapting inferences, and finding final answers to problems. Thus, students are expected to engage in the production of analogies, and teachers seem to systematically ensure that they participate. Even so, the components students are asked to generate often do not necessitate consideration of the structural similarities and alignment between source and target objects and thus do not require deep analogical reasoning.

Due to the nature of students' participation, there was little information available to determine whether students were performing like participants in laboratory contexts and failing to spontaneously build inferences from analogies with low surface similarity or whether they were benefitting and would be able to transfer the analogies to additional contexts. This uncertainty creates a problem for teachers using analogies. Although they may believe they are assessing the students' comprehension by asking them to participate in production of the analogy, the students are not required to demonstrate analogical reasoning to perform the type of participation requested. Thus, teachers are not gathering relevant information about their students' comprehension of the analogies.

In sum, although teachers are not gathering optimal online assessments of their students' learning from analogy, they are generating sophisticated structural analogies on a regular basis. They are manipulating source and target selection according to mathematical and student contextual information, and they are doing so in ways that mirror empirical findings regarding beneficial analogy practices. Thus, consideration of context seems integral to the reasoning process of generating analogies for these teachers. However, teachers may be failing to provide an important learning opportunity for students by maintaining control over the reasoning process rather than distributing the thinking more substantially for each analogy.

ACKNOWLEDGMENTS

The TIMSS data analyzed for this study was collected within the video portion of the Third International Mathematics and Science Study supported by the National Center of Education Statistics (NCES), U.S. Department of Education, and the National Science Foundation (NSF). We are grateful to Vandana Thadani, Ann Fink, Krysten Peterson, and Erin Dougherty for hours of coding assistance.

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