

Hummel, J. E., & Holyoak, K. J. (2001). A process model of human transitive inference. In M. Gattis (Ed.), *Spatial Schemas in Abstract Thought* (pp. 279-305). Cambridge, MA: MIT Press.

A Process Model of Human Transitive Inference

John E. Hummel Keith J. Holyoak

University of California, Los Angeles

John E. Hummel
Department of Psychology
University of California, Los Angeles
405 Hilgard Ave.
Los Angeles, CA 90095-1563
jhummel@lifesci.ucla.edu

This research was supported by NSF Grant 9729023.

If you are told that job A pays more than job B and job B pays more than job C, then you can conclude that job A pays more than job C. This conclusion follows deductively because *pays-more-than()*—or, more generally, *more-than()*—is a transitive relation. How does the human mind reason about transitive relations such as *more-than()*, *taller-than()*, etc.? Transitive relations are logically well-behaved: Given that some relation, *R()*, is transitive, and given the facts *R(A, B)* and *R(B, C)*, we can conclude *R(A, C)* with certainty. As such, there are many ways in principle to reason about them. For example, it is easy to imagine a cognitive "module" devoted to transitive reasoning. Like a program written in PROLOG, the module might take statements about transitive relations as input—for example, statements of the form *pays-more-than(A, B)* and *pays-more-than(B, C)*—and generate various inferences—such as *pays-most(A)*, *pays-least(C)*, and *pays-more-than(A, C)*—as output. However, the behavioral evidence suggests that this is not how the human mind processes transitive relations. A logical module would not provide any inherent basis for predicting that some transitive inferences will be more difficult to make than others. But it has been known for a long time that people solve some transitive-inference problems more quickly and accurately than others (e.g., Clark, 1969; DeSoto, London & Handel, 1965; Huttenlocher, 1968; McGonigle & Chalmers, 1977, 1984).

As suggested by several other chapters in this volume, the machinery of visuospatial reasoning may provide an economical basis for reasoning about transitive relations. Our visual systems are adept at computing spatial relations—such as *above()*, *larger-than()*—and many of these relations are transitive: If object A is above object B and B is above C, then A will be above C. Importantly, the visual machinery that computes these relations from the information in a visual image must have this knowledge built into it implicitly. The reason is that images are quintessentially analog: If, in some image, A is above B and B is above C, then A will necessarily be above C, so the same machinery that computes A above B and B above C (from their locations in the image) also has the information necessary to compute A above C. To the machinery that computes spatial relations from visual images, the "inference" that A is above C is not an inference at all, but rather a simple observation. This is not to say that visuospatial reasoning is the only basis—or even the phylogenetically or ontogenetically "first" basis—for reasoning about transitive relations (see McGonigle & Chalmers, this volume, for a thorough review of alternative approaches; also McGonigle & Chalmers, 1977, 1984), but it does provide an efficient potential basis for solving such problems.

For this reason, numerous researchers have proposed that people reason about transitive relations by exploiting the properties of spatial arrays (e.g., Huttenlocher, 1968). The general hypothesis is that, given statements such as "A is more than B" and "B is more than C", we map A, B, and C onto locations in a spatial array (e.g., with A near the top, B in the middle and C at the bottom). Given this mapping, the machinery that computes spatial relations from visual images can easily compute that A is above (in this case more than) C, that A is at the top (in this case is most), and that C is at the bottom (least). There is empirical support for the use of a mental array to make transitive inferences in solving three-term series problems (e.g., DeSoto et al., 1965; Huttenlocher, 1968; see also McGonigle & Chalmers, 1984), and the mental array hypothesis is also supported by the inverse distance effect commonly found when people make comparative judgments in series of four to 16 items (e.g., Potts, 1972, 1974; Woocher, Glass & Holyoak, 1978). That is, the further apart two items are in an ordered series, the faster people can decide which is greater or lesser. The distance effect is readily accounted for by perceptual discriminability of items in an array (Holyoak & Patterson, 1981), or in a monotonic ordering (McGonigle & Chalmers, this volume), but has no ready explanation in terms of logical inference rules. The mental array hypothesis is also appealing for its simplicity and economy: Given that we come equipped with routines for processing transitive relations in the domain of visual perception, it seems natural to assume that these same routines might be recruited in aid of reasoning about transitive relations in non-visual domains. But in spite of the appeal of the general mental array hypothesis, a detailed account of exactly how these operations might work has remained elusive (but see Byrne & Johnson-Laird, 1989, for a process model of a related task).

The task of developing an adequate process model of transitive inference is made much more difficult by the need to incorporate mental operations that go beyond the use of a spatial array.

In particular, there is clear evidence that some of the variations in the difficulty of transitive inferences are due to linguistic factors, such as negation and markedness of comparative adjectives (e.g., the marked form *worse* in premises tends to lead to slower solutions than the unmarked or neutral form *better*; see Clark, 1969; Sternberg, 1980). To accommodate such findings as well as those that support use of a spatial array, Sternberg (1980) proposed a mixed spatial-linguistic model. This model provides a good mathematical fit to data on solution times for a range of transitive inference problems, but it is not a process model. Developing a process model of transitive inference is further complicated by evidence indicating that transitive reasoning depends on limited-capacity working memory. For example, the abilities of preschool children to make transitive inferences appear to be subject to capacity limitations (Halford, 1984), and transitive inference can be greatly impaired by damage to the prefrontal cortex, the apparent neural substrate for the form of working memory required for integrating relations (Waltz et al., 1999). It thus appears that transitive inference requires both working memory resources and linguistic representations, neither of which are specifically spatial. More generally, mapping from linguistic statements of premises to spatial representations, and mapping back from spatial relations to concrete inferences about specific objects (i.e., verbally storable conclusions), requires the machinery of visuospatial processing to communicate with other parts of the cognitive apparatus. Thus, even if visuospatial processing plays a role in transitive reasoning, it cannot be the whole story (cf. McGonigle & Chalmers, this volume). How might non-visual representations and processes communicate with the machinery of visuospatial processing for transitive reasoning, and how might the resulting algorithms give rise to the behavioral properties of human transitive reasoning? These are the questions to which the current chapter is addressed.

Visuospatial Routines for Transitive Inference

Using visuospatial routines for transitive reasoning entails solving at least two problems. First, it is necessary to map objects onto locations in a spatial array based on their pair-wise relations. Second, it is necessary to use the resulting values (locations) to compute additional relations. The second problem—computing categorical relations based on locations in an array—is relatively straightforward, and to solve it we will borrow (and adapt) a set of routines that Hummel and Biederman (1992) developed to compute the spatial relations among object parts for the purposes of object recognition.

The first problem—mapping objects to locations based on their pair-wise relations—is more challenging. First, it is under-constrained: Knowing that A is more than B and B is more than C is not sufficient to specify any unique assignment of values to A, B and C. A system for mapping objects onto values based on their pair-wise relations must therefore be prepared to make "guesses" about the specific locations of objects in the array. A second challenge concerns the semantics of transitive relations. If transitive relations are to serve as the basis for mapping objects onto a *spatial* array, then the mental representation of non-spatial transitive relations must somehow capture what they have in common with more obviously spatial relations such as *above()* and *larger()*. That is, the "language of thought" must make the mapping from non-spatial relations (such as *more-than()*, *better-than()* or *smarter-than()*) onto spatial predicates transparent, and the cognitive architecture must be configured to exploit this language. Although this latter constraint is obvious, it is less obvious how to satisfy it. Atomic symbols of the type used by traditional symbolic models of cognition do not make the relationship between different predicates explicit. In order to solve the object-to-value mapping problem, we will borrow from the literature on analogical mapping, augmenting Hummel and Holyoak's (1997) LISA⁰ model of analogical reasoning with a set of routines for mapping pair-wise relations onto locations in a spatial array.

LISA's algorithm for analogical mapping forms the basis of the current model of transitive reasoning, so we will start by reviewing how that algorithm works. Next, we will describe the "Mental Array Module" (MAM), a set of routines that augment LISA by allowing it to convert pair-wise categorical relations into hypotheses about the numerical values (i.e., locations) of specific objects, and to compute additional relations based on those values. The resulting system (which we will simply refer to as "LISA" for brevity's sake) is a process model of human transitive reasoning. Given statements such as "A is more than B" and "B is more than C", LISA can answer questions such as "Which is greatest?" and "Which is least?". And like the human, LISA solves some problems faster than others. We will conclude by presenting LISA's simulations of a large body of data on human transitive reasoning (specifically, the data from 16 of the kinds of problems that Sternberg, 1980, administered to human subjects).

The LISA Model of Analogical Thinking

LISA is an integrated model of the major stages of analogical thinking, which include access (retrieving a useful source analog from long-term memory given a novel target problem or situation as a cue), mapping (discovering the correspondences among the elements of the source and target), analogical inference (using the source to make inferences about the target), and schema induction (using the source and target together to induce a more general schema or rule). For the purposes of the current model, only LISA's mapping algorithm is important. We review that algorithm in detail below. For a more complete discussion of access, inference and schema induction in LISA, see Holyoak and Hummel (in press) and Hummel and Holyoak (1997).

Human thinking is both structure-sensitive and flexible. It is structure-sensitive in the sense that we can represent and reason about abstract relationships, appreciating for example the similarities and differences between the idea "John loves Mary" and the idea "Mary loves John" (see Fodor & Pylyshyn, 1988). It is flexible in the sense that we can tolerate imperfect matches. For example, we effortlessly understand how "John loves Mary" is similar to (and different from) "Bill loves Susan". Together, the structure-sensitivity and flexibility of human thinking constitute

⁰"LISA" for "Learning and Inference with Schemas and Analogies".

challenging design requirements for a model of the human cognitive architecture. Traditional symbolic models easily capture the structure-sensitivity of human mental representations, but do not naturally capture their flexibility. The proposition *loves* (John, Mary) clearly specifies who loves whom, and makes transparent the structural relations between the ideas "John loves Mary" and "Mary loves John". But it fails to specify what John and Mary have in common and how they differ.⁰ It likewise fails to specify that *loves* (John, Mary) is semantically more similar to *loves* (Bill, Susan) than to *loves* (Bill, Rover). Traditional connectionist representations have the opposite strengths and weaknesses. Distributed representations of Mary, Susan and Rover can specify what these objects have in common and how they differ (i.e., capture their semantic content), but traditional connectionist models lack provisions for binding simple representations, such as feature vectors describing Bill and Rover, into symbolic relational structures, such as *loves* (Bill, Rover) (see Fodor & Pylyshyn, 1988; Holyoak & Hummel, in press; Hummel & Holyoak, 1997; Marcus, 1998).

The core of the LISA model is a basis for dynamically binding distributed (i.e., connectionist) representations of relations and objects into structured (i.e., symbolic) representations in working memory (WM), and using those representations for memory retrieval, analogical mapping, inference and schema induction. (See Holyoak & Hummel, in press, for a review of connectionist approaches to role-filler binding.) LISA uses synchrony of firing for dynamic binding in WM (Hummel & Holyoak, 1992; Shastri & Ajjanagadde, 1993). Case roles and objects are represented as patterns of activation on a collection of *semantic units* (small circles in Figure 1); units representing case roles and objects fire in synchrony when they are bound together and out of synchrony when they are not. Because these representations are distributed, they naturally capture the semantic content of the predicates and objects they describe; and because they bind the roles of predicates dynamically to their fillers, they also capture the symbolic structure of the proposition as a whole. The resulting representations simultaneously capture both the flexibility and structure-sensitivity of human representations of propositions.

Every proposition is encoded in LTM by a hierarchy of *structure units* (Figure 1). At the bottom of the hierarchy are *predicate* and *object* units. Each predicate unit locally codes one case role of one predicate. For example, *love1* represents the first ("lover") role of the predicate "love", and has bi-directional excitatory connections to all the semantic units representing that role (e.g., *emotion1*, *strong1*, *positive1*); *love2* represents the second ("beloved") role and is connected to the corresponding semantic units (e.g., *emotion2*, *strong2*, *positive2*). Semantically-related predicates share units in corresponding roles (e.g., *love1* and *like1* share many units), making the semantic similarity of different predicates explicit. Object units are just like predicate units except that they are connected to semantics describing things rather than roles. For example, the object unit *Mary* might be connected to units for *animal*, *human*, *adult*, and *female*, whereas *Rover* might be connected to *animal*, *dog*, *pet*, and *male*.

Sub-proposition units (SPs) bind roles to objects in LTM. For example, *love* (John Mary) would be represented by two SPs, one binding John to lover (*love1*), and the other binding Mary to beloved (*love2*) (see Figure 1). The *John+love1* SP has bi-directional excitatory connections with *John* and *love1*, and the *Mary+love2* SP has connections with *Mary* and *love2*. *Proposition* (P) units reside at the top of the hierarchy and have bi-directional excitatory connections with the corresponding SP units. P units serve a dual role in hierarchical structures (such as "Sam knows that John loves Mary"), and behave differently according to whether they are currently serving as the "parent" of their own proposition or the "child" (i.e., argument) of another (see Hummel & Holyoak, 1997). Structure units do not encode semantic content in any direct way. Rather, they serve only to store that content in LTM, and to generate (and respond to) the corresponding synchrony patterns on the semantic units.

⁰Traditional symbolic models typically capture the semantic content of symbols through external devices, such as look-up tables and semantic networks. In this approach, symbols serve as "pointers" to semantic content, rather than capturing that content directly (see Hinton, 1990; Hummel & Holyoak, 1997).

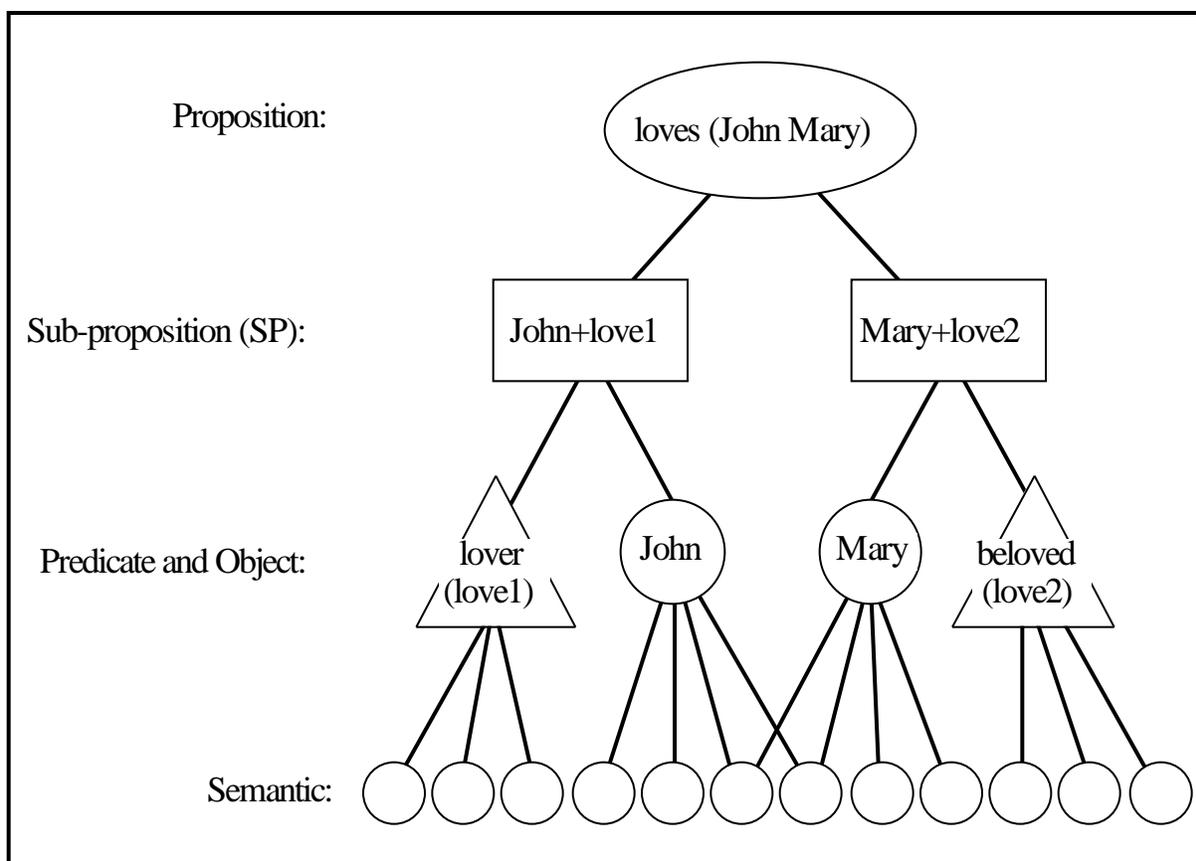


Figure 1. Illustration of the representation of a single proposition in LISA's LTM. The oval is a proposition (P) unit, rectangles are sub-proposition (SP) units, triangles are predicate units, large circles are object units, and small circles are semantic units. See text for details.

An analog (e.g., episode, problem or story) is represented as collections of structure units coding the propositions in that analog (see Figure 2). Separate analogs do not share structure units, but do share the same semantic units. The final component of LISA's architecture is a set of *mapping connections* between structure units of the same type in different analogs. Every P unit in one analog shares a mapping connection with every P unit in every other analog; likewise, SPs share connections across analogs, as do objects and predicates. For the purposes of mapping and retrieval, analogs are divided into two mutually exclusive sets: a *driver* and one or more *recipients*. Retrieval and mapping are controlled by the driver. (There is no necessary linkage between the driver/recipient distinction and the more familiar source/target distinction.) LISA performs mapping as a form of guided pattern matching. As P units in the driver become active, they generate (via their SP, predicate and object units) synchronized patterns of activation on the semantic units (one pattern for each role-argument binding): Semantic units for roles fire in synchrony with the semantic units coding their fillers, and separate role-filler bindings fire out of synchrony with one another. Because the semantic units are shared by all analogs, the patterns generated by a proposition in one analog will tend to activate one or more similar propositions in other analogs in LTM (analog retrieval) or in WM (analogical mapping).

Mapping differs from retrieval solely by the addition of the modifiable mapping connections. During mapping, the weights on the mapping connections grow larger when the units they link are active simultaneously, permitting LISA to learn the correspondences generated during retrieval. By the end of a simulation run, corresponding structure units will have large positive weights on their mapping connections, and non-corresponding units will have strongly negative weights. These connection weights also serve to constrain subsequent access and mapping: If object A maps to object B in one context, then the mapping connection thereby acquired will

encourage it to map to B in future contexts. This property plays a critical role in LISA's ability to integrate multiple relations, appreciating for example, that *Jim* in *taller*(Sam Jim) is the same object as *Jim* in *taller*(Jim Bill).

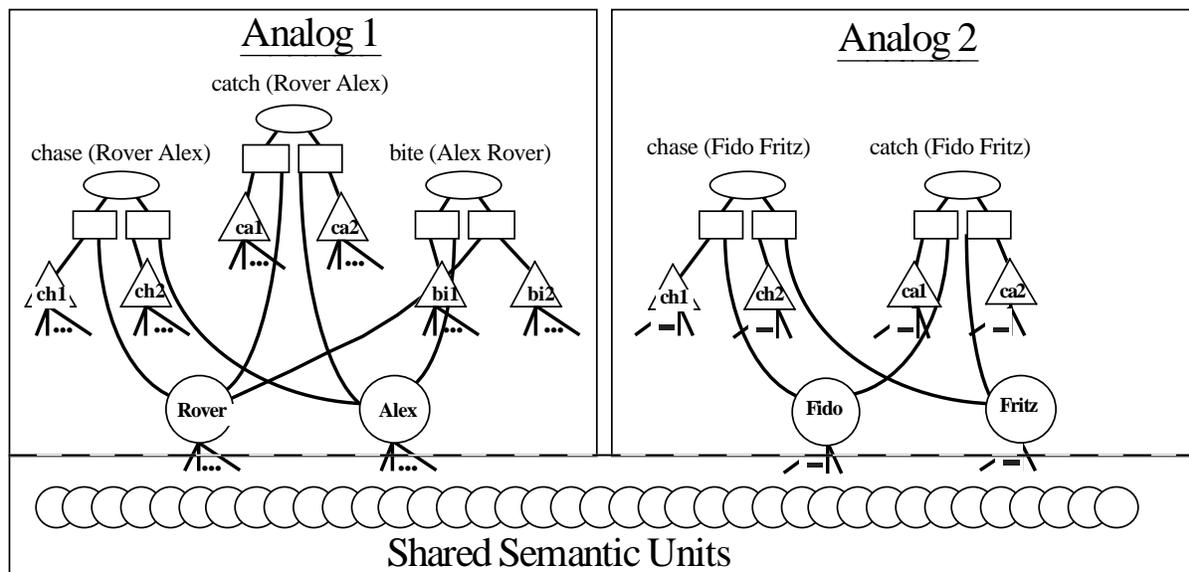


Figure 2. Illustration of the representation of whole analogs in LISA's long-term memory. Analog 1 states that Rover chased Alex, Rover caught Alex, and Alex bit Rover. Analog 2 states that Fido chased Fritz and Fido caught Fritz. Each proposition is coded by a collection of structure units (as in Figure 1). Within an analog, a single structure unit codes a single object, predicate, or proposition regardless of how many times it is mentioned (e.g., in Analog 1, the same object unit represents Rover in all three propositions). Separate analogs do not share structure units but do share semantic units.

Mapping in LISA is bootstrapped by shared semantics in objects and predicates across analogs, but an important property of LISA's mapping algorithm is that it permits objects and predicates to map even when they have no semantic overlap. For example, imagine that LISA is mapping "John loves Mary" in the driver onto "Susan loves chocolate" in the recipient, and that the representation of Mary shares no semantic features at all with the representation of chocolate. LISA will still map Mary onto chocolate by virtue of their shared binding to the *loved* (*love2*) role of a *loves* relation. When the SP *love2+Mary* fires in the driver, the semantics activated by *love2* will activate *love2* in the recipient. *Love2* will activate the SP *forlove2+chocolate* (in the recipient), which will activate *chocolate*. As a result, *Mary* (in the driver) will be active at the same time as *chocolate* (in the recipient), so LISA will strengthen the weight on the mapping connection between them. This ability to map dissimilar objects based on shared relational roles (or to map dissimilar roles based on similar arguments) plays a key role in LISA's ability to map arbitrary objects to spatial locations based on their pair-wise relations.

Mapping Objects onto a Spatial Array: The Mental Array Module (MAM)

Given a set of pair-wise relations among a collection of objects, our current goal is to use those relations to map the objects onto unique locations in a spatial array. For example, given the propositions *taller*(Bill, Joe) and *taller*(Joe, Sam), the goal is to map Bill to a location near the top of the array, Joe to a location near the middle, and Sam to a location near the bottom. To accomplish this mapping, LISA is equipped with a Mental Array Module (MAM; see Figure 3), a specialized module for manipulating visuospatial relations. Like any other analog in LISA, MAM contains a collection of object and predicate units, which have connections to the semantic units that are shared by all analogs. In MAM, predicate units represent the roles of the *greater-than* relation (which we will designate "greater-subject", *gS*, and "greater-referent", *gR*) and the *less-than*

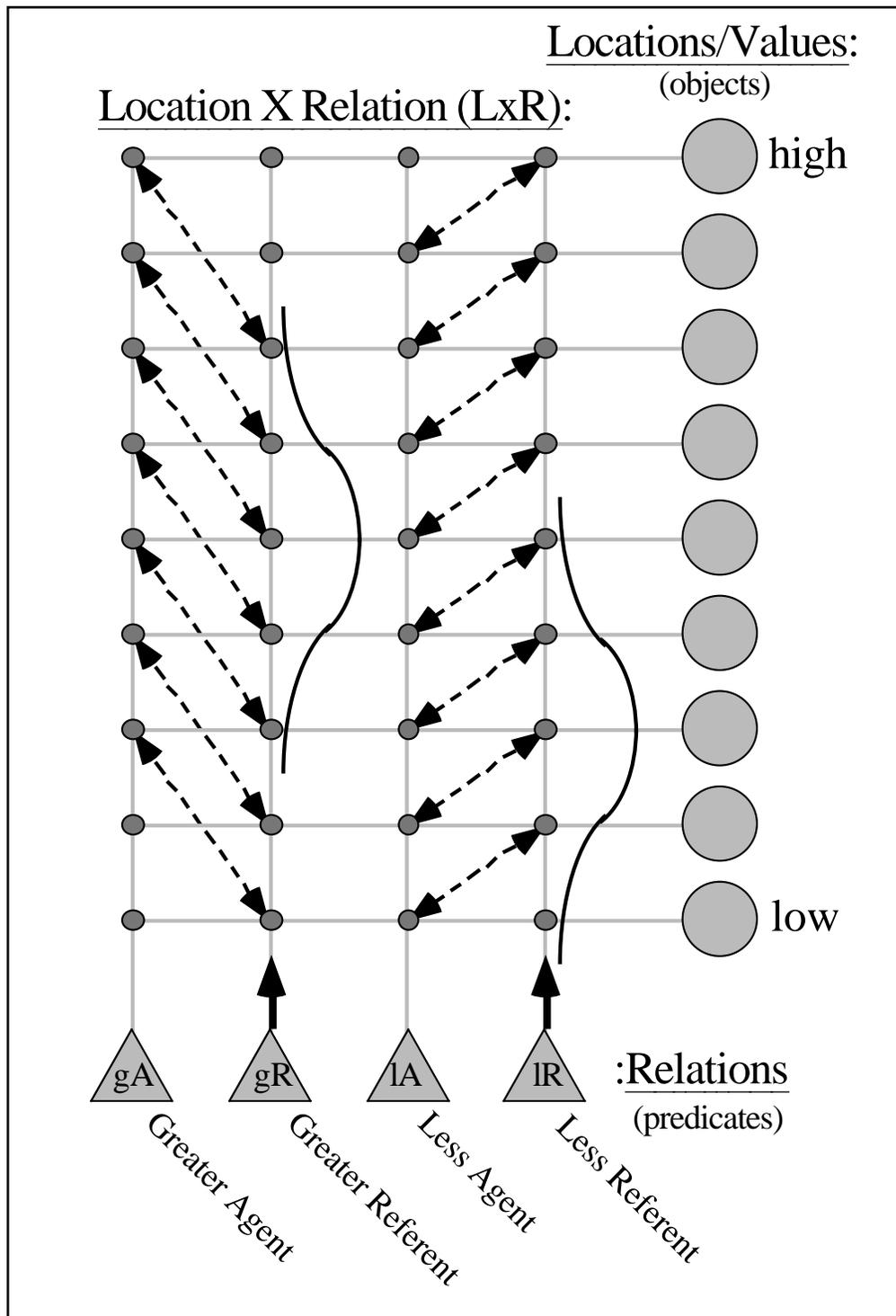


Figure 3. LISA's "Mental Array Module" (MAM). Object units (large circles) represent locations (values) in the mental array and predicate units (triangles) represent the roles of the relations *greater-than* and *less-than*. Location-by-Relation (LxR) units mediate the communication between location units and relation units. See text for details.

relation (less-subject, *lS*, and less-referent, *lR*). Object units represent locations (values) in a spatial array.⁰ In contrast to an "ordinary" analog, MAM also contains an array of *location-and-relation*, *LxR*, units (Hummel & Biederman, 1992), each of which represents a conjunction of one relational role at one location in the array. For example, the *LxR* unit *6-gS* represents any object whose location (value) is 6 and is greater than the location (value) of some other object. Each *LxR* unit has excitatory connections to the corresponding relation and location units (horizontal and vertical gray lines in Figure 3), and mediates the communication between them. These units make it possible to compute categorical relations (such as *greater-than* (A, B)) given specific values (such as A = 6 and B = 4) as input, and vice versa.

Representation. Using MAM to compute categorical relations from specific values is straightforward, and is not the primary focus of the current effort (for details, see Hummel & Biederman, 1992). The reverse mapping—from categorical relations to specific values—is more challenging. Given only the knowledge that A is greater than B, there is no unique assignment of A and B to values in the array. In order to bootstrap the mapping from categorical relations to specific values, MAM contains two additional sets of connections, which amount to heuristic "guesses" about the likely value of an object based on the relational role(s) to which it is bound. First, the referent relations, *gR* and *lR*, activate their corresponding *LxR* units in a Gaussian pattern (Gaussian curves in Figure 3). For example, *gR*, the referent role of the greater-than relation, strongly excites the *LxR* unit corresponding to *greater-referent at location 5* (*5-gR*), and more weakly excites those surrounding location 5 (the strength of excitation falls off according to a Gaussian function). This pattern of excitation corresponds to the assumption: Given no knowledge about an object except that it is the referent of a greater-than relation, assume that object has a medium value. (Location 5 is the center of the array.) Similarly, *lR*, the referent role of the less-than relation, excites *LxR* units in a Gaussian pattern centered over location 3 (below the center). These connections correspond to the assumption: Given no knowledge except that an object is the referent of a less-than relation, assume that the object has slightly less than a medium value. Whereas "greater-than" is an unmarked relation, "less-than" is the marked form of the relation. We assume that markedness manifests itself in the assumption that an object has a smaller-than-average value. For example, told that Sam is taller than Fred (the unmarked form), LISA assumes that Fred (the referent) is of average height; but told that Fred is shorter than Sam, it assumes that Sam (the referent) is shorter than average (see Clark, 1969). The assumption that marked adjectives are selectively mapped onto the lower range of the array is a key representational assumption about the connection between linguistic and spatial representations of comparatives.

The second set of additional connections in MAM (dashed diagonal arrows in Figure 3) capture the logical dependencies among the referent and subject roles of the greater-than and less-than relations. *LxR* units representing the referent role of greater-than excite all *LxR* units representing the subject role of the greater-than relation at values two *above* themselves. For example, *LxR* unit *5-gR* (greater-referent at location 5) excites unit *7-gS* (greater-subject at location 7), and vice-versa (i.e., the connections are reciprocal): Whatever the value, V_r , of the object bound to the referent role of the greater-than relation, the value, V_s , of the object bound to the subject role must be greater than that (i.e., $V_s > V_r$). Conversely, *LxR* units representing the referent role of the less-than relation excite *LxR* units for the subject role of that relation one unit *below* themselves, and vice versa: Whatever the value, V_r , of the object bound to the referent role of a less-than relation, the value, V_s , of the object bound to the subject role must be less than that (i.e., $V_s < V_r$). Markedness manifests itself in these connections as the distance between connected *LxR* units

⁰In the current implementation, the array contains nine locations/values [units], and for simplicity location is coded in a linear fashion (e.g., the distance between the locations coded by the 8th and 9th units is the same as the distance between the locations coded by the first and second). Although this linear representation of number is not psychologically realistic, it is sufficient for our current purposes, and there are no a priori reasons to expect that adapting the algorithm to work with a nonlinear representation of number would change its properties in any substantive way.

representing the unmarked relation relative to the distance between LxR units representing the marked relation. The former is twice the latter, corresponding to the assumption that the arguments of an unmarked relation are likely to be less similar than the arguments of a marked relation: Told that Fred is shorter than Sam (the marked form), LISA will assign Fred and Sam to more similar heights than when told that Sam is taller than Fred. This assumption, which plays an important role in LISA's ability to simulate the findings of Sternberg (1980) (as elaborated shortly), follows as a natural consequence of the fact that the referent role of a marked relation assumes a lower value than the referent role of an unmarked relation: Because *lR* activates a value below the mean of the array, and because *lS* must activate a still lower value, the two roles are "crowded" together near the bottom of the array; as a result, the distance between *lR* and *lS* is necessarily smaller than that between *gR* and *gS* (which, by virtue of *gR*'s placement at the center of the array, are not crowded together near the top; see Figure 3).

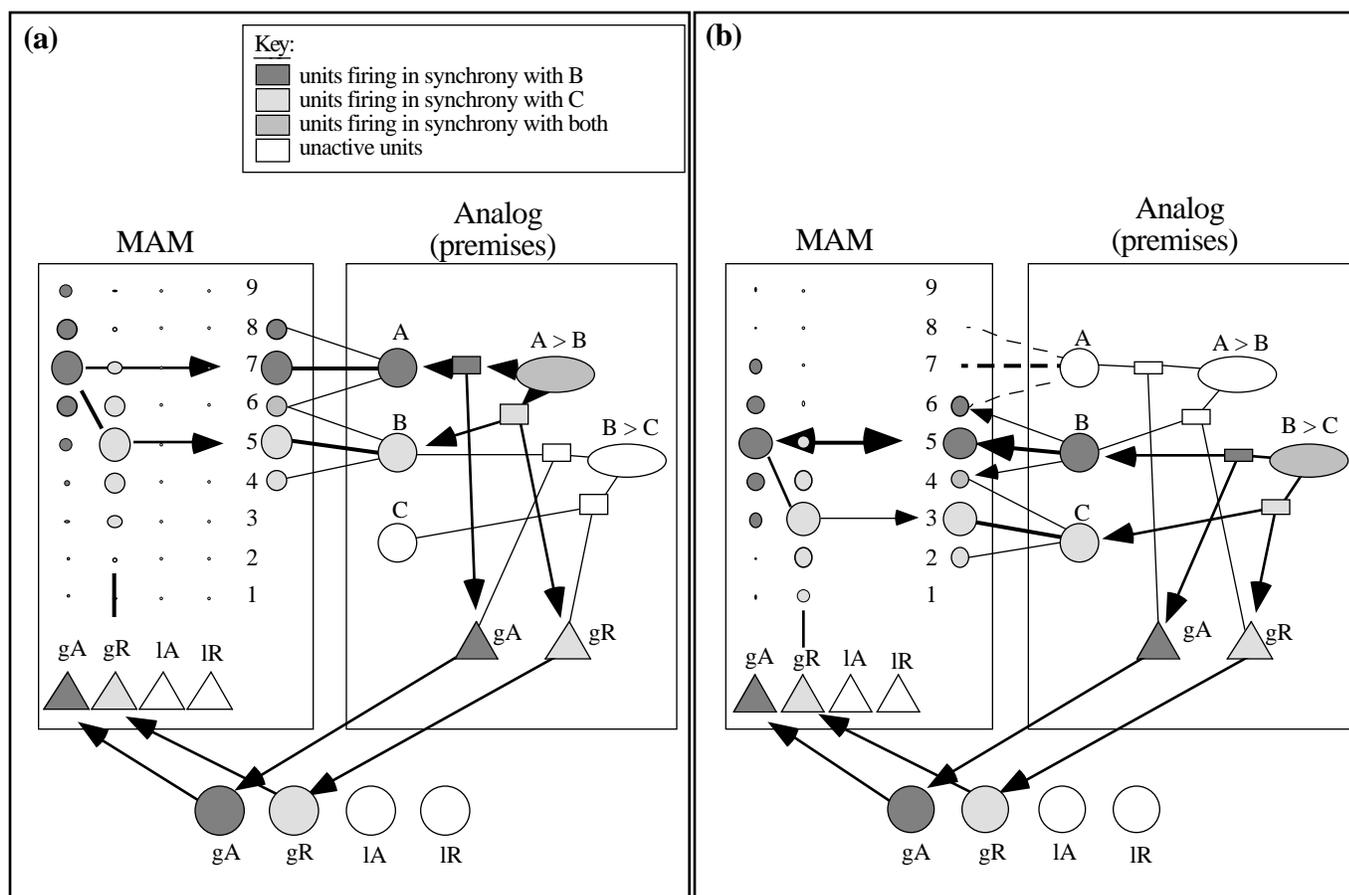


Figure 4. Illustration of the operation of MAM. (a) When the proposition *greater-than* (A B) fires in the problem, MAM activates location 7 in synchrony with A (i.e., A maps to location 7) and location 5 in synchrony with location B (B maps to location 5). (b) When *greater-than* (B C) fires in the problem, B maps to location 5 (by virtue of the mapping connection established previously), so C maps to location 3.

Operation. The interactions among the location, relation and LxR units cause LISA to assign objects to locations based on their categorical relations. Imagine that we tell LISA that A is greater than B and B is greater than C, as illustrated in Figure 4. ("Telling" LISA these facts means creating an analog containing the propositions *greater-than* (A, B) and *greater-than* (B, C), and designating that analog as the driver.) When the proposition *greater-than* (A, B) becomes active, *greater-subject+A* will fire out of synchrony with *greater-referent+B* (Figure 4a). When *greater-referent+B* fires, the predicate unit *greater-referent* (in Analog 1) will excite the semantic unit *greater-referent*, which in turn will excite *gR* (*greater-referent*) in MAM. Inside MAM, *gR* will

excite LxR units in a Gaussian pattern centered at location 5. In turn, these units will excite LxR units for greater-subject centered at location 7 (recall that greater-referent LxR units excite greater-subject LxR units two values above themselves; see Figure 4a). The output from LxR units to location units is gated by the activation of the corresponding relation unit. As a result, when *gR* fires (which it will do in synchrony with *greater-referent* and therefore object B in the problem), the LxR units for greater-referent will activate the location units surrounding location 5; as a consequence, Object B in the problem will map to (i.e., develop a positive mapping weight to) the object unit in MAM representing location 5. When greater-subject (*gS*) fires in MAM (which it will do in synchrony with *greater-subject* and therefore object A in the problem), it will allow the greater-subject LxR units to activate location units surrounding location 7; Object A in the problem will therefore map to the object unit in MAM representing location 7 (Figure 4a).

In the course of mapping *greater-than* (A, B) to MAM, LISA thus maps object A onto location 7 and object B onto location 5, and stores these mappings as weights on the corresponding mapping connections. These connections serve as linking relations, allowing LISA to integrate the first proposition, *greater-than* (A, B), with the second, *greater-than* (B, C), based on their shared argument, B. Specifically, when the proposition *greater-than* (B, C) becomes active, *greater-subject+B* will fire out of synchrony with *greater-referent+C* (Figure 4b). *Greater-referent* (in the analog) will tend to cause C to map to location 5, just as it caused B to map to location 5 in the context of *greater-than* (A, B). However, this time, object B already has an excitatory mapping connection to location 5, so when *greater-subject+B* fires, object B will activate location 5 directly (i.e., by way of the mapping connection; Figure 4b). In turn, location 5 will activate LxR units for location 5 in the greater-subject role. Recall that in the proposition *greater-than* (B, C), B is bound to (i.e., synchronized with) the *subject* role of the greater-than relation. The greater-subject LxR units will activate greater-referent LxR units two values below themselves, i.e., centered around location 3 (recall that greater-referent LxR units activate greater-subject LxR two units above themselves, and because the connections are reciprocal, greater-subject units activate greater-referent units two values *below* themselves). The result is that, with object B activating location 5, object C will tend to map to location 3.

After both *greater-than* (A, B) and *greater-than* (B, C) have had the opportunity to fire a few times, object A will map strongly to location 7, B to location 5 and C to location 3. LISA will thus have mapped the objects to locations in a spatial array based on their pair-wise relations. As detailed shortly, these locations serve as the basis for LISA's discovery of other relations among the objects, including the fact that A is greatest, C is least, and A is greater than C. It is important to emphasize the role of the linking relation between B in the first proposition and B in the second (embodied in the mapping connection from B to location 5) in this mapping. Without the ability to learn this connection, LISA would map A to location 7 and B to location 5 in the context of *greater-than* (A, B), and map B to location 7 and C to location 5 in the context of *greater-than* (B, C). The resulting mappings would be useless for inferring any additional relations among the objects.

The example in Figure 4, based on the premises *greater-than* (A, B) and *greater-than* (B, C), was chosen for ease of illustration. Similar principles apply when the relations among A, B and C are stated differently (e.g., *less-than* (B, A) and *less-than* (C, B), or *less-than* (C, B) and *greater-than* (A, B)). However, as illustrated in the context of the specific simulations, not all premises are equally easy for LISA to process. For example, problems based on the less-than relation tend to be slightly more difficult for LISA to solve than problems based on greater-than. Recall that the marked form, less-than, tends to push the objects closer together and lower in the spatial array. This makes it more difficult for MAM/LISA to decide which is greatest or which is least. LISA and MAM are also sensitive to other factors, such as the order in which the premises are stated.

Computing Greatest and Least

While LISA and MAM are mapping objects to locations based on their pair-wise relations, MAM also keeps track of the mounting evidence regarding which objects occupy the greatest and smallest locations/values. Each location unit, i , in MAM is associated with an integrator, I_i , which accumulates evidence from other location units, j , above and below itself. At each instant in time, t , I_i is updated according to:

$$I_i^t = \sum_{j < i} A_j^t - 0.8 \sum_{j > i} A_j^t, \quad (1)$$

where A_j^t is the activation of location unit j at time t . Whenever a location lower than i (i.e., $j < i$) fires (i.e., $A_j^t > 0$), I_i is incremented, adding to the evidence that any object at location i (if there is one) has the greatest value; whenever a location higher than i (i.e., $j > i$) fires, I_i is decremented, subtracting from the evidence that any object at location i has the greatest value, and (equivalently) adding to the evidence that any object at location i has the smallest value. Negative evidence (i.e., from $j > i$) accumulates more slowly than positive evidence, as indicated by the 0.8 in Eq. (1). As a result, it takes LISA/MAM longer to decide which value is the smallest than it takes to decide which is greatest.

Equation (1) describes how each location unit accumulates evidence about whether it represents the greatest or smallest value by monitoring whether there are active locations above or below itself. As time progresses, the integrator on the highest location will accumulate a progressively larger (i.e., more positive) value, and the integrator on the lowest location will accumulate a progressively larger negative value. At each instant, t , a global integrator, Γ , monitors the values of all individual integrators, I_i , by computing a gated sum of their values:

$$\Gamma^t = \sum_i I_i^t A_i^t 0.2|i - 4.5|. \quad (2)$$

Effectively, the value of the global integrator at any instant, t , is the value of the local integrator (I_i) associated with whichever location, i , happens to be active at instant t , weighted by the distance between i and a fixed point (location 4.5) just below the center of the spatial array. For example, if object A is mapped to location 7, and object A is active at time t , then location 7 will be active at time t . At time t , the value of the global integrator will be the value of the integrator on location 7, multiplied by the distance between location 7 and location 4.5. If LISA is seeking the object with the greatest value, and if Γ^t is greater than a threshold value (+2700), then processing stops and LISA declares whichever object is currently active (i.e., the object firing in synchrony with location i —in the above example, object A) to have the greatest value. If LISA is seeking the object with the smallest value, and if Γ^t is lower than a threshold value (-2700), then processing stops and LISA declares whichever object is currently active to have the smallest value. Weighting Γ^t by the distance between i and the center of the array biases LISA to respond "greatest" fastest to objects mapped to very high values, and "smallest" fastest to objects mapped to very small values.

Simulations

We evaluated LISA/MAM as a model of human transitive inference by testing it on a subset of the problems Sternberg (1980) gave his subjects. We chose these data as the basis of our simulations because they represent a complete sampling of ways to state transitive inference problems (as detailed below), and are based on a large number of subjects to ensure reliability of the response-time data. In a series of four experiments, Sternberg gave subjects problems of the general form "Bill is better than Joe. Joe is better than Sam. Who is best?", and recorded how long it took them to solve them. The problems varied on four dimensions:

(1) The order and markedness of the stated premises. As illustrated in the table below, both premises could be stated in the unmarked form (UU; e.g., "Bill is better than Joe; Joe is better than Sam"), both could be stated in the marked form (MM: "Joe is worse than Bill; Sam is worse than Joe"), the first could be unmarked and the second marked (UM: "Bill is better than Joe; Sam is worse than Joe"), or the first could be marked and the second unmarked (MU: "Joe is worse than Bill; Joe is better than Sam").

<u>Condition</u>	<u>First</u>	<u>Second</u>	<u>Example</u>
UU	unmarked	unmarked	"Bill is better than Joe. Joe is better than Sam."
MM	marked	marked	"Joe is worse than Bill. Sam is worse than Joe."
UM	unmarked	marked	"Bill is better than Joe. Sam is worse than Joe."
MU	marked	unmarked	"Joe is worse than Bill. Joe is better than Sam."

(2) Negation of the premises. The premises could either be stated as negations (e.g., "Joe is not as good as Bill"), or as non-negated predicates ("Bill is better than Joe").

(3) The question asked. Subjects were asked to specify either the top-most object (e.g., "Who is best?") or the bottom-most object ("Who is worst?").

(4) Presentation order. Premises were either stated in a *forward* order, so that the highest pair of objects was mentioned first (e.g., in "Bill is better than Joe; Joe is better than Sam", Bill and Joe, both of whom are higher than Sam on the dimension of goodness, are mentioned first), or the *reverse* order, so that the lowest pair of objects was mentioned first (e.g., "Joe is better than Sam; Bill is better than Joe").

In all, Sternberg recorded subjects' response times and accuracy in 32 different conditions: Markedness and order (4) X Negation (2) X Question (2) X Presentation order (2). Of these, we will consider the 16 non-negated conditions obtained by crossing Markedness and order X Question X Presentation order. (The data in the 16 negated conditions are similar to those in the 16 non-negated conditions, except that negation adds roughly a constant to subjects' response times; see Sternberg, 1980.) Some of Sternberg's conditions involved precuing, in which either the two premises or the question were presented prior to the possible answers (people's names). We only simulated data from the uncued conditions, in which all the problem information was presented at once (premise 1, premise 2, question, and alternative names, respectively, on successive vertically-arranged lines on a tachistoscopically-presented card). The data in the 16 non-negated uncued conditions are summarized below. Numbers indicate the rank order of mean response time across conditions, averaged over Sternberg's four experiments, with 1 being the shortest (fastest) and 16 the longest (slowest).

Table 1: Sternberg (1980) Data
(non-negated conditions)

		Presentation Order	
		Forward	Backward
Question	Top-most ("Who is best?")	UU (3) MM (14) UM (7) MU (10)	UU (1) MM (8) UM (2) MU (4)
	Bottom-most ("Who is worst?")	UU (4) MM (12) UM (6) MU (8)	UU (10) MM (16) UM (13) MU (14)

The data are complex, but several major trends are apparent. First, there is an interaction between presentation order and question asked, such that the reverse order (in which the premise containing the lower pair of items is stated first) is easier than the forward order (stating the upper pair first) when the subject must report the top-most object (e.g., "Who is best?"), and harder than the forward order when the subject must report the bottom-most object ("Who is worst?"). This pattern can be viewed as a kind of recency effect: Responses are faster when the correct answer is contained in the second premise stated (in the lower premise in the case of "Who is least?", and in the upper in "Who is greatest?"). Second, when both premises have the same markedness status (i.e., UU or MM), responses are faster in the unmarked conditions than in the marked conditions (i.e., UU is faster than MM in all four cells of Table 1). And third, in the mixed-markedness conditions (UM and MU), responses are faster when the upper pair is stated in the unmarked form than when the upper pair is stated in the marked form (i.e., UM is faster than MU in all four cells of Table 1).

Simulation Procedure

We tested LISA on the 16 conditions in Table 1. Premises were stated in the simple form *greater-than* (x, y) (unmarked) and *less-than* (x, y) (marked). We varied the premises (i.e., conditions UU, MM, UM, and MU) by varying which predicate (greater-than or less-than) was used to describe the upper pair of objects (A/B), and which was used to describe the lower pair (B/C). For example, condition MU stated the propositions *less-than* (B, A) (upper pair marked) and *greater-than* (B, C) (lower pair unmarked); UM stated *greater-than* (A, B) and *less-than* (C, B). We varied the presentation order by manipulating whether the upper or lower pair was presented first. In the "forward" order, the proposition describing the higher pair (H; objects A and B) was presented first, and in the reverse order, the proposition describing the lower pair (L; objects B and C) was presented first. Specifically, in the forward order, H was presented⁰ twice, then L was

⁰ "Presenting" a proposition to LISA entails activating it and allowing its SP, object and predicate units to fire for 600 iterations (see Hummal & Holyoak, 1997).

presented once, followed by H, then L twice, resulting in the ordering: H H L H L L. We created the reverse order simply by switching the higher and lower pairs in the ordering: L L H L H H. These orderings reflect our assumption that the subject first thinks about the first premise stated (e.g., H H, in the forward order), then transitions to the second (L H), and finally concentrates on the second (L L). We asked LISA "Who is greatest?" by setting the threshold on the global integrator, Γ , to +2700; to ask "Who is least?" we set the threshold to -2700. A run terminated as soon as Γ crossed the threshold, and we recorded LISA's response time (i.e., the number of iterations run to solution). Although a single run entails up to three presentations of each proposition, LISA was usually able to answer the question before all six propositions had run. (If LISA was unable to answer the question after all six propositions had run, then it continued to run propositions in a random order until it could answer the question.) We ran each condition ten times. The data reported below are means over the ten runs.

Simulation Results

Figure 5 shows a scatter plot of LISA's mean response time in each condition (in iterations) against the mean response times of Sternberg's (1980) subjects in the same conditions. The correlation coefficient (Pearson's r) between LISA's response times and those of the human subjects is .82, or $r^2 = .67$. Sternberg reported fits of r^2 ranging from .34 to .51 across individual experiments between subjects' response times for uncued problems and his most successful mathematical model, the mixed linguistic-spatial model. Our LISA simulations thus compare favorably with the fits provided by Sternberg's models. As the parameter space for LISA is not comparable to that for Sternberg's mathematical models, and we did not perform any systematic search for a "best fit" with LISA, the relative fits are simply suggestive that LISA can account for the human pattern of data at least as well as the most successful previous model. It should also be noted that we simulated only 16 of Sternberg's 32 data points (omitting the 16 data points from the negated conditions), so LISA's correlation with the human data suffers from range restriction. This range restriction would be expected to reduce LISA's fit given that the negated vs. non-negated dimension accounts for a large proportion of the variance in Sternberg's data (see Sternberg, 1980). But the most important distinction between LISA and Sternberg's models is that LISA is a process model. That is, in contrast to Sternberg's models, LISA actually *solves* the problems and so provides an algorithmic account of how people may solve them. Sternberg's models are strictly mathematical models, the only function of which is to predict the response times themselves. Mathematical models cannot actually solve the problems, and hence provide little insight into the likely psychological algorithms underlying response times to achieve solutions.

Table 2 shows the rank ordering of LISA's response times by condition alongside the corresponding data from Sternberg's (1980) subjects. LISA captures most of the major trends in the human data. First, it captures the interaction between presentation order and question asked: As in the human data, the reverse order is faster than the forward order when the question is "Who is greatest?", and slower than the forward order when the question is "Who is least?". Second, when both premises have the same markedness status (i.e., UU or MM), LISA's responses are faster in the unmarked conditions than in the marked conditions (compare UU to MM in corresponding cells of Table 2). The one exception is in the Backward-Least condition, where LISA reversed the relative ease of the MM and UU conditions. Third, recall that for Sternberg's subjects in the mixed-markedness conditions (UM and MU), problems were easier when the upper pair was stated in the unmarked form than when the upper pair was stated in the marked form (i.e., for humans, UM is faster than MU in all four cells of Table 2). LISA shows this trend in the forward condition but not in the backward condition.

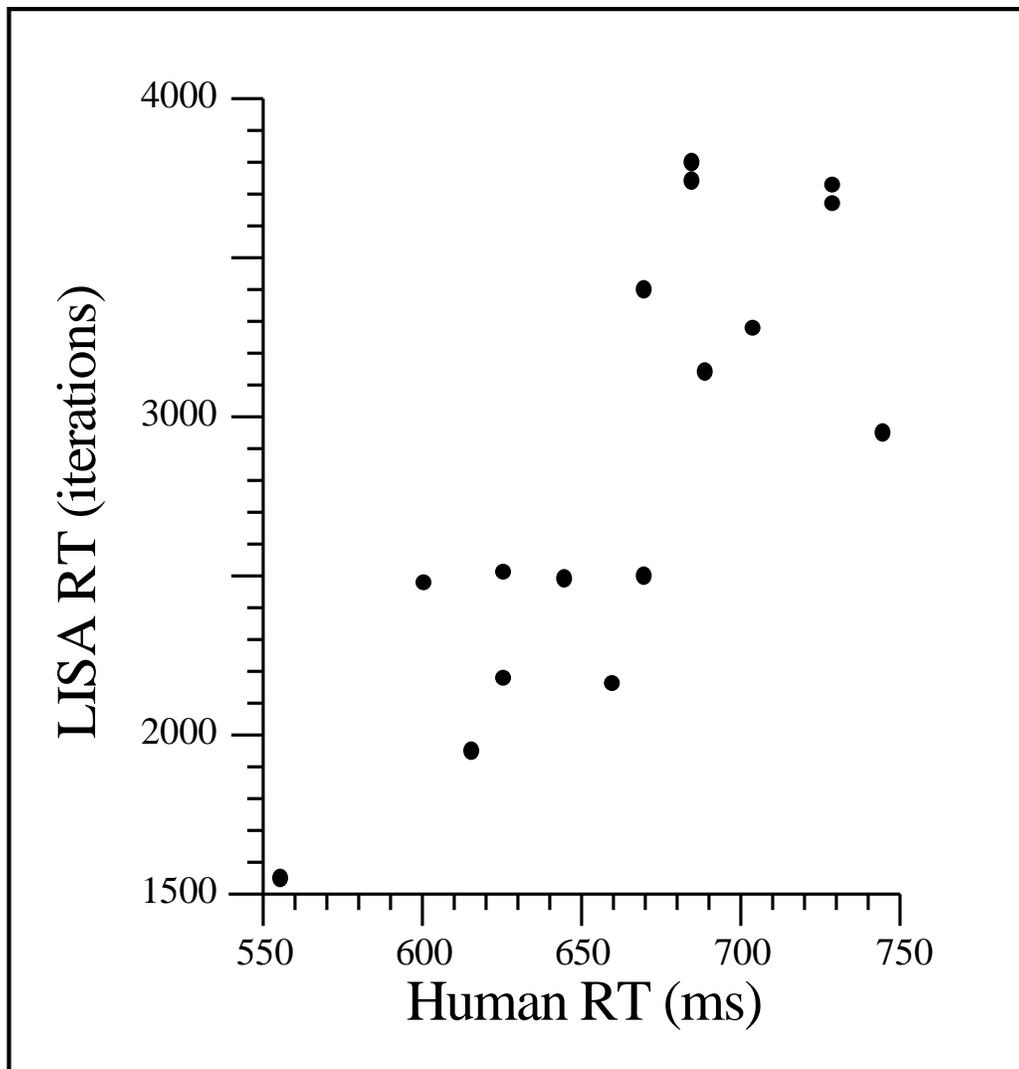


Figure 5. LISA's response times in 16 of Sternberg's (1980) conditions plotted against human response times in the same conditions (human response times from Sternberg, 1980).

Discussion

We have described a process model of human transitive reasoning that integrates the LISA model of analogical reasoning (Hummel & Holyoak, 1997) with routines for computing relations borrowed from Hummel and Biederman's (1992) model of shape perception and object recognition. The model provides the first explicit algorithmic account of mechanisms by which an internal spatial array (e.g., as used for computing spatial relations in visual perception) might be used to perform transitive inferences. By integrating a Metric Array Model, MAM, within the LISA architecture, the augmented LISA provides an account of how such a spatial module might interact with general semantic representations of word meanings and with the working-memory limits that constrain human reasoning. Rather than strictly segregating linguistic and spatial aspects of transitive inference (as in Sternberg's, 1980, mixed model), LISA shows how the meanings of comparatives such as *greater* and *lesser* can be specified in terms of the operation of the spatial module itself. One consequence of this integration of linguistic and spatial representations is that the effects of markedness (e.g., the relative difficulty of problems stated in terms of *lesser* rather than *greater*) arise from the mapping between the meanings of the comparatives and the spatial positions in MAM, rather than from strictly linguistic processes.

**Table 2: LISA Data and Sternberg (1980) Data
(non-negated conditions)**

		Presentation Order				
		Forward		Backward		
Question	Top-most ("Who is most?")	<u>Human</u>	<u>LISA</u>	<u>Human</u>	<u>LISA</u>	
		UU	(3)	(2)	UU	(1) (1)
		MM	(14)	(14)	MM	(8) (7)
		UM	(7)	(3)	UM	(2) (5)
		MU	(10)	(16)	MU	(4) (4)
	Bottom-most ("Who is least?")	<u>Human</u>	<u>LISA</u>	<u>Human</u>	<u>LISA</u>	
	UU	(4)	(8)	UU	(10) (15)	
	MM	(12)	(10)	MM	(16) (9)	
	UM	(6)	(6)	UM	(13) (11)	
	MU	(8)	(12)	MU	(14) (13)	

It is interesting to note that the internal array MAM uses to represent quantities, magnitudes, and locations is not any sort of literal "image" of objects in specific locations: To represent A at the top of the array, B in the middle and C at the bottom, MAM does not generate picture-like depictions of A, B and C on any sort of mental viewing screen. Rather, it is sufficient to bind symbolic representations of the objects (object units in the architecture of LISA) to symbolic representations of their values or magnitudes (object units in MAM corresponding to specific locations). Indeed, for the purposes of computing the relations among objects, this symbolic representation of magnitude is superior to any literal image-based representation of objects in specific locations (see Hummel & Biederman, 1992). Like the representation the Hummel and Biederman model uses to compute spatial relations, the symbolic representation of magnitude in MAM specifies all and only that which is relevant to the relation of interest—namely, the values or magnitudes that need to be compared. A literal image, by contrast, would specify (implicitly or explicitly) all manner of irrelevant information, including the size and font in which a letter is written, whether it is upper or lower case, and so on. (Note that it is impossible to draw a picture of an A without drawing some specific A.) None of these properties are relevant to the task of deciding whether the A is above or below the C. (Of course, it is an empirical question whether the representations underlying human visuospatial reasoning are as "uncluttered" by irrelevant details; the current model predicts that they will be.) This point is worth emphasizing because the contrast between analog versus discrete representations is often confused with the contrast between image-based versus symbolic representations (the assumption being that all analog representations are image-based, and all propositional—i.e., symbolic—representations must be discrete). MAM, like the visual routines from which it is derived, is an example of an analog representation that is nonetheless decidedly symbolic (see Holyoak & Hummel, in press, for a discussion of LISA as a symbolic system).

We speculate that most or all of the visual representations that make contact with higher cognition must have this property. For example, we believe the present approach could be extended to account for data concerning how people make order judgments with prestored arrays (Potts,

1972, 1974), and how they determine the relative magnitudes of concepts stored in semantic memory, such as digit magnitudes (Moyer & Landauer, 1967) and animal sizes (Moyer, 1973). In addition to routines that perform transitive inferences, routines that reason on the basis of visual relations in domains such as graph interpretation (Gattis & Holyoak, 1996; Gattis, this volume) and sign language (Emorey, this volume), may depend on symbolic (i.e., propositional) structural descriptions that make explicit what it is necessary to know and discard the rest. (See Hummel, in press, for similar arguments about the nature of object recognition and Pylyshyn, 1973, for similar arguments about the nature of visual imagery.)

LISA also provides an account of why working memory (and in particular, the working memory associated with prefrontal cortex) is essential for transitive inference. Waltz et al. (1999) showed that patients suffering from degeneration of frontal cortex perform at chance on transitive inference problems that depend on the capacity to integrate multiple relations. For example, in order to conclude that Bill is taller than Fred given the premises "Sam is taller than Fred" and "Bill is taller than Sam," the reasoner must integrate the first premise with the second based on their common reference to Sam.⁰ In contrast to frontal patients, age- and IQ-matched control patients with degeneration in anterior temporal cortex (which is presumably not as directly implicated in relational integration), perform transitive inference tasks as well as age-matched controls (with near-perfect accuracy). These and other findings strongly suggest that frontal cortex must be centrally involved in the kind of working memory that is responsible for relational integration (see Waltz et al., 1999).

LISA/MAM provides a straightforward account of such findings in terms of the role of the mapping connections in the ability to integrate relations. Recall that the mapping connections play a central role in LISA's ability to integrate *greater-than* (A, B) with *greater-than* (B, C) for the purposes of mapping A, B and C to unique locations in the spatial array: It is the connection from B to location 5 (established when *greater-than* (A, B) fires) that makes it possible to keep B at location 5 and place C at location 3 when *greater-than* (B, C) fires (see Figure 4b). Without this connection, B would map to location 7 and C to location 5 (in the context of *greater-than* (B, C)) just as A goes to 7 and B to 5 in the context of *greater-than* (A, B). The resulting representation would be useless for deciding which object is greatest and which least. And indeed, run with the mapping connections disabled, LISA, like frontal patients, performs randomly on the transitive inference task.

In this context, it is tempting to speculate that the mapping connections of LISA may be realized neurally as neurons in prefrontal cortex (see also Hummel & Holyoak, 1997). This account makes additional (and to our knowledge, novel) predictions about the role of prefrontal cortex in other reasoning tasks. One is that frontal patients should perform poorly on analogies requiring relational integration. For example, consider the following simple analogy:

<u>Story 1</u>	<u>Story 2</u>
Bill proposed to Mary.	Joe proposed to Sally.
Bill bought a Chevy.	Joe bought a Ford.
	Fred bought a Buick.

It is not difficult to realize that Bill maps to Joe rather than Fred because they both proposed marriage to their sweethearts. Based on this mapping, it is also clear that the Chevy (which Bill bought) must map to the Ford (which Joe bought) rather than the Buick (which Fred bought). But the mapping of the Chevy to the Ford is specified only because of the "proposal" statements, so discovering it requires the reasoner to integrate "Joe proposed to Sally" with "Joe bought a Buick" based on their common reference to Joe. LISA's account of transitive reasoning predicts that

⁰By contrast, if problems are consistently stated in the order, "Bill is taller than Sam" and "Sam is taller than Fred", then it is possible to find the solution using various chaining strategies that do not require relational integration, but also do not generalize to arbitrary orderings. Patients suffering from degeneration of frontal cortex solve problems stated in this canonical ordering as well as age-matched controls (see Waltz et al., 1999).

frontal patients should have as much difficulty discovering this simple mapping as they have reasoning about transitive relations.

It seems very likely that many other aspects of human reasoning, besides transitive inference, also depend on the integration of semantic knowledge and working-memory operations with representations derived from those that support visuospatial perception. We hope the model we have described here may provide an example of how the connections between perception and thought may be given explicit realization in a neural architecture.

References

- Byrne, R. M. J., & Johnson-Laird, P. N. (1989). Spatial reasoning. *Journal of Memory and Language*, 28, 564-575.
- Clark, H. H. (1969). Linguistic processes in deductive reasoning. *Psychological Review*, 76, 387-404.
- DeSoto, C., London, M., & Handel, S. (1965). Social reasoning and spatial paralogic. *Journal of Personality and Social Psychology*, 2, 513-521.
- Emorey, K. (this volume).
- Fodor, J. A., & Pylyshyn, Z. (1988). Connectionism and cognitive architecture: A critical analysis. *Cognition*, 28, 3-71.
- Gattis, M. (this volume).
- Gattis, M., & Holyoak, K. J. (1996). Mapping conceptual to spatial relations in visual reasoning. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, 22, 231-239.
- Hinton, G. E. (1990). Mapping part-whole hierarchies into connectionist networks. *Artificial Intelligence*, 46, 47-75.
- Halford, G. S. (1984). Can young children integrate premises in transitivity and serial order tasks? *Cognitive Psychology*, 16, 65-93.
- Holyoak, K. J., & Hummel, J. E. (in press). The proper treatment of symbols in a connectionist architecture. In E. Deitrich & A. Markman (Eds.), *Cognitive dynamics: Conceptual change in humans and machines*. Mahwah, NJ: Erlbaum.
- Holyoak, K. J., & Patterson, K. K. (1981). A positional discriminability model of linear order judgments. *Journal of Experimental Psychology: Human Perception and Performance*, 7, 1283-1302.
- Hummel, J. E., & Biederman, I. (1992). Dynamic binding in a neural network for shape recognition. *Psychological Review*, 99, 480-517.
- Hummel, J. E., & Holyoak, K. J. (1992). Indirect analogical mapping. In *Proceedings of the Fourteenth Annual Conference of the Cognitive Science Society* (pp. 516-521). Hillsdale, NJ: Erlbaum.
- Hummel, J. E., & Holyoak, K. J. (1997). Distributed representations of structure: A theory of analogical access and mapping. *Psychological Review*, 104, 427-466.
- Huttenlocher, J. (1968). Constructing spatial images: A strategy in reasoning. *Psychological Review*, 75, 550-560.
- Kosslyn, S. M. (1980). *Image and mind*. Cambridge, MA: Harvard University Press.
- Marcus, G. F. (1998). Rethinking eliminative connectionism. *Cognitive Psychology*, 37, 243-282.
- Maybery, M. T., Bain, J. D., & Halford, G. S. (1986). Information-processing demands of transitive inference. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, 12, 600-613.
- McGonigle, B. O., & Chalmers, M. (this volume).
- McGonigle, B. O., & Chalmers, M. (1977). Are monkeys logical? *Nature*, 267, 694-696.
- McGonigle, B. O., & Chalmers, M. (1984). The selective impact of question form and input mode on the symbolic distance effect in children. *Journal of Experimental Child Psychology*, 37, 525-554.
- Moyer, R. S. (1973). Comparing objects in memory: Evidence suggesting an internal psychophysics. *Perception & Psychophysics*, 13, 180-184.
- Moyer, R. S., & Landauer, T. K. (1967). Time required for judgments of numerical inequality. *Nature*, 215, 1519-1520.
- Potts, G. R. (1972). Information processing strategies used in the encoding of linear orderings. *Journal of Verbal Learning and Verbal Behavior*, 11, 727-740.
- Potts, G. R. (1974). Storing and retrieving information about ordered relationships. *Journal of Experimental Psychology*, 103, 431-439.
- Pylyshyn, Z. (1973). What the mind's eye tells the mind's brain: A critique of mental imagery. *Psychological Bulletin*, 80, 1-24.
- Sternberg, R. J. (1980). Representation and process in linear syllogistic reasoning. *Journal of Experimental Psychology: General*, 109, 119-159.
- Waltz, J. A., Knowlton, B. J., Holyoak, K. J., Boone, K. B., Mishkin, F. S., de Menezes Santos, M., Thomas, C. R., & Miller, B. L. (1999). A system for relational reasoning in human prefrontal cortex. *Psychological Science*, 10, 119-125.
- Woocher, F. D., Glass, A. L., & Holyoak, K. J. (1978). Positional discriminability in linear orderings. *Memory & Cognition*, 6, 165-173.