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Modeling Discrete and Continuous Entities With Fractions and Decimals

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When people use mathematics to model real-life situations, their use of mathematical expressions is often mediated by *semantic alignment* (Bassok, Chase, & Martin, 1998): The entities in a problem situation evoke semantic relations (e.g., tulips and vases evoke the functionally asymmetric "contain" relation), which people align with analogous mathematical relations (e.g., the noncommutative division operation, tulips/vases). Here we investigate the possibility that semantic alignment is also involved in the comprehension and use of rational numbers (fractions and decimals). A textbook analysis and results from two experiments revealed that both mathematic educators and college students tend to align the discreteness versus continuity of the entities in word problems (e.g., marbles vs. distance) with distinct symbolic representations of rational numbers—fractions versus decimals, respectively. In addition, fractions and decimals tend to be used with nonmetric units and metric units, respectively. We discuss the importance of the ontological distinction between continuous and discrete entities to mathematical cognition, the role of symbolic notations, and possible implications of our findings for the teaching of rational numbers.

Keywords: math cognition, education, fractions, decimals, semantic alignment, symbolic notation

How do people understand abstract mathematic concepts? How do they select the appropriate mathematical concepts to solve real-life problems? To help students achieve both goals, mathematics educators use "word problems"—short stories describing simple real-life situations involving various entities that can be modeled by the target mathematical concepts. For example, the mathematical concept of a fraction is often illustrated with a word problem describing a pizza that is shared by several children. The pizza is sliced into *n* equal slices, and each slice is denoted by the fraction *1/n*. Note that in order to be effective as examples of the target mathematical concepts, the situations described in the word problems, or "situation models," have to be analogous to their mathematical representations, or "mathematical models" (Kintsch & Greeno, 1985). For example, in the above pizza problem, the mathematical concept of a fraction requires that the pizza slices be equal in size.

Semantic Alignment in Understanding Mathematical Problems

People who have extensive experience with solving word problems are highly systematic in selecting mathematical models that correspond to the situation models (e.g., Bassok, Chase, & Martin, 1998; Bassok, Wu, & Olseth, 1995; Dixon, 2005; Dixon, Deets, & Bangert, 2001; Mochon & Sloman, 2004; Sherin, 2001; Waldmann, 2007). But how do students and mathematics educators decide that particular situations are analogous to particular mathematical models? Bassok et al. (1998) have proposed that such modeling decisions are guided by *semantic alignment*. The thrust of the semantic-alignment process is that the entities in a problem situation elicit semantic relations (e.g., tulips and vases are likely to evoke the functionally asymmetric "contain" relation), which people then align with structurally analogous mathematical relations (e.g., the noncommutative division operation, tulips/vases). Both children and adults find it easier and more natural to solve or construct semantically aligned rather than misaligned word problems (e.g., tulips/vases rather than tulips/roses; Martin & Bassok, 2005), and for many adults the process of semantic alignment is highly automatic (Bassok, Pedigo, & Oskarsson, 2008; Fisher, Bassok, & Osterhout, 2010).

In addition to semantic inferences about object relations, the entities in word problems elicit inferences about the continuity versus discreteness of these entities, which then affect people's modeling decisions. To illustrate, a word problem that describes constant change in the value of a coin evokes a situation model of continuous change, whereas a word problem that describes constant change in salary evokes a situation model of discrete changes. These distinct situation models lead college students to generate qualitatively distinct solutions to otherwise mathematically isomorphic word problems (Bassok & Olseth, 1995). The distinction between continuous and discrete concepts also influences the interpretation of graphs and diagrams. For example, line graphs tend to be interpreted as representing changes in continuous variables, whereas bar graphs tend to be interpreted as representing differences among levels of discrete variables (Shah, Mayer, & Hegarty, 1999; for reviews see Shah & Hoeffner, 2002; Hegarty &

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Stull, 2012). Gestures also show the influence of quantity type. People produce corresponding continuous "swipe" or discrete "tap" gestures when describing problems based on continuous versus discrete variables (Alibali, Bassok, Olseth, Syc, & Goldin-Meadow, 1999).

The contrast between discreteness and continuity, which is captured by the linguistic distinction between count and mass nouns (e.g., marbles vs. water, respectively; see Bloom, 1994; Bloom & Wynn, 1997), is a basic ontological distinction that affects how people parse the world. For example, Spelke, Breinlinger, Macomber, and Jacobson (1992) have argued that young babies use this distinction to discriminate between objects: Continuity of motion indicates that a single object is moving in space, whereas discontinuity indicates the existence of more than one object. Importantly, this distinction plays a crucial role in the development of "number sense" (Dehaene, 1997). According to Dahaene and his colleagues, an approximate sense of magnitude for mass entities (i.e., a continuous variable) is evolutionarily more primitive than exact calculations with discrete objects (Feigenson, Dehaene, & Spelke, 2004; Dehaene, 1997).

Children eventually acquire counting procedures, which constitute a critical basis for the development of number concepts related to discrete objects (see Rips, Bloomfield, & Asmuth, 2008 for a review, discussion and commentaries). Counting, or "enumerating" (Gelman & Gallistel, 1978), is the first opportunity for children to explicitly align entities with numbers. The counting process involves one-to-one mapping of consecutive integers to distinct objects (e.g., stickers, chairs), such that a child increments the integer magnitude simultaneously with the act of moving through the set of objects, with the last number denoting the set cardinality (3 stickers, 4 chairs). The use of integers in counting discrete objects precedes their use in exact quantifications of continuous entities (2 lbs of sugar, 3 feet), which require explicit parsing of mass entities into countable measurement units (Miller, 1984; Mix, Huttenlocher, & Levine, 2002a; Boyer, Levine, & Huttenlocher, 2008; Nunes, Light, & Mason, 1993).

Alignment of Discrete and Continuous Entities With Fractions and Decimals

Prior research has documented that the solutions of word problems reflect semantic alignments between the continuity and discreteness of situation and mathematical models. The present set of studies aimed to examine whether people treat numerical notations themselves as mathematical models of discrete and continuous entities. Whereas integers can be readily aligned with either discrete or discretized continuous entities, representing *parts* of such entities requires the use of rational numbers, notated as either fractions or decimals (e.g., [1/2] of the marbles, 0.5 L of water). The studies we report in the present paper applied the semantic-alignment framework to examine whether the discreteness versus continuity of the entities that appear in word problems affects people's tendency to represent parts or proportions of these entities with different mathematical symbols for rational numbers—fractions versus decimals, respectively.

Fractions and decimals are two distinct notations for numbers. Some fraction can represent any rational number, whereas some decimal (unbounded in length) can represent any real number (where the rational numbers are a subset of the reals). When decimals are bounded (as they are in all experimental work, for obvious reasons), they cannot exactly express the magnitudes of all real (or rational) numbers (e.g., 1/3), but can approximate them closely.

Fractions and decimals are typically used as alternative notations for the same magnitude, other than rounding error (e.g., 3/8 km vs. 0.375 km). For example, the Common Core State Standards Initiative (2014) for Grade 4 refers to decimals as a "notation for fractions." However, there are conceptual differences between the two notations that could affect their alignment with parts or portions of discrete and continuous entities (see Figure 1). A fraction represents the ratio formed between the cardinalities of two sets, each expressed as an integer; its bipartite format (a/b) captures the value of the part (the numerator *a*) and the whole (the denominator *b*). A decimal can represent the one-dimensional magnitude of a fraction ($a/b = c$) expressed in the standard base-10 metric system. Whereas a fraction represents a two-dimensional relation, a corresponding decimal represents a one-dimensional magnitude (English & Halford, 1995; Halford, Wilson, & Phillips, 1998) in which the variable denominator of a fraction has been replaced by an implicit constant (base 10).

The fraction format is well-suited for representing sets and subsets of discrete entities (e.g., balls, children) that can be counted and aligned with the values of the numerator (*a*) and the denominator (*b*) (e.g., *3/7* of the balls are red). Also, as is the case with integer representations, the fraction format can be readily used to represent continuous entities that have been discretized—parsed into distinct equal-size units—and therefore can be counted (e.g., 5/8 of a pizza).[1] In contrast, the one-dimensional decimal representation of such discrete or discretized entities seems much less natural (~*0.429* of the balls are red; *0.625* of a pizza), and may suggest partition of nondivisible entities (e.g., balls).

The decimal format is well-suited to represent portions of continuous entities, particularly because unbounded decimals capture all real numbers (i.e., all points on a number line). This alignment is likely to be especially strong when decimals (base 10) are used to model entities that have corresponding metric units (*0.3* meters, *0.72* liters). When continuous entities have nonmetric units (e.g., imperial measures with varied bases such as 12 in. or 60 minutes), their alignment with decimals may require computational transformations. Given that the denominator of a fraction is a variable that can be readily adapted to any unit base, it may be computationally easier to represent nonmetric measures of continuous entities with fractions (*2/3* of a foot) than with decimals (*0.67* ft). Because computational ease may interact with the natural conceptual alignment of continuous entities with decimals, we predict that metric units should be predominantly represented with decimals, whereas imperial units may be represented by fractions.

The above analysis suggests that semantic knowledge about the discreteness or continuity of entities in word problems will lead people to select either fractions or decimals as symbolic mathematical models of these entities, with unit type (imperial or metric) playing a secondary role (yielding an especially strong affinity between decimals and continuous variables measured in metric

[1] Note that "3/4 of the sandwiches" is very natural because each sandwich in a set is a discrete object; "3/4 of the sandwich" requires imagining that a single sandwich has been divided into four equal parts.

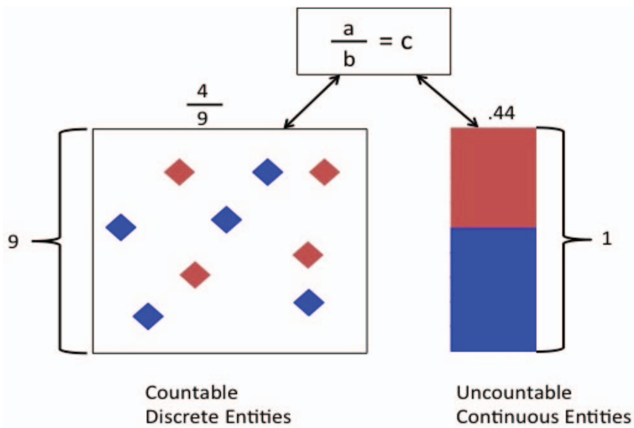

Figure 1. Hypothesized alignment of fractions and decimals with discrete and continuous entities. See the online article for the color version of this figure.

units). In the set of studies reported below, we provide evidence that educators and highly educated adults adhere to this hypothesized alignment when using fractions and decimals to describe discrete and continuous entities, respectively. The first study was an analysis of all the word problems involving fractions or decimals in a commonly used textbook series (Grades K through 8). These problems were constructed by math educators who, one would assume, were aiming to help students understand rational numbers by providing them with situation models that could be modeled by these numbers. After describing the results of the textbook analysis, we report results from two experiments in which undergraduate students constructed word problems involving either fractions or decimals (Experiment 1), and selected continuous or discrete diagrams to represent combinations of rational numbers and unit types (Experiment 2).

Textbook Analysis

We examined the set of word problems mathematics educators present to students as situation models of rational numbers, coding the entities (discrete or continuous) that are modeled by fractions and decimals.

Method

Materials and Design

We examined the *Addison-Wesley Mathematics* (1989) textbook series from grades kindergarten through 8th grade. This particular textbook series was chosen because it is representative of the mathematics teaching that most current college undergraduates would have received in their early education (i.e., during the 1990s). This textbook series has historically had a large market share (20–25%) and has been widely used across the United States (Mix, Levine, & Huttenlocher, 1999). Subsequent versions of this series (titled *enVision* and *Realize*, published by Pearson) remain in widespread use today. The K-8 grade levels were selected because they cover the main introduction and use of rational numbers in math curricula before the start of formal algebra. We analyzed all the problems that involved rational numbers, a total of 874 problems (504 with fractions, 370 with decimals).

Problem Coding

We developed a coding scheme that categorized problems by their number type (*fraction* vs. *decimal*) and entity type (*continuous* vs. *countable*). Problems were classified as *fraction* or *decimal* based on the number type that appeared in the problem text or were called for in the answer. Problems were selected that only contained decimals, or else only contained fractions, enabling us to separately classify each rational number type. Because we were not evaluating answers to the problems, if the problem called for an answer in a particular rational number type, this was not included in the coding scheme. Problems were classified as *continuous* or *countable* based on the entities in the problems. Continuous problems involved entities that are referred to linguistically as "mass nouns" (e.g., weight, volume, length). Importantly, these continuous entities were treated as wholes (e.g., the length of a string) and were not explicitly broken down into smaller countable pieces (as in a string that was cut into three equal parts). We also coded the unit type used in the continuous problems (yes/no base-10) in order to assess whether the base-10 format of decimals is used more often with readily aligned base-10 units than with nonbase-10 units.

Countable problems involved either discrete or explicitly discretized entities. Discrete entities were sets of individual objects that cannot be broken down into natural equal units (e.g., marbles, balloons, or grapes). Continuous entities that were parsed into equal countable parts (e.g., an apple cut into equal slices, or a rectangle divided into equal squares) were coded as "discretized." In addition, the discretized category encompassed collective nouns (e.g., people, class), which are collections of countable nouns (a person, a student). Collective nouns thus refer to a mass with meaningful, discrete units; hence we included collective nouns with other masses (e.g., apple slices) that are portioned into meaningful units. Examples of the coded problems appear in Table 1.

One research assistant coded all of the problems using the above coding scheme. In order to assess interrater reliability, a second researcher coded a random sample of 350 problems (i.e., 40% of the total problems). The second coder was blind to the original coder's judgments. The two coders agreed on 336 (96%) of the sampled textbook problems. A third researcher, who was blind to the first two coders' judgments, then coded the 14 problems on which the first two coders had differed. These problems were then placed into whichever category it was assigned by two of the three coders.

Results and Discussion

The results of the survey of textbook problems are shown in Figure 2. Of the 874 total problems, 504 used fractions and 370 used decimals. Continuous entities comprised a large majority of the decimal problems (78%). In a complementary way, countable entities comprised a majority of the fractions problems (57%). A chi-square test of independence between number type and continuity confirmed that the two factors were significantly associated ($\chi^2(2, N = 874) = 115.7, p < .001$).

A significant portion of the continuous-entities problems involved currency (for decimal problems, $n = 101$; for fraction

Table 1
Examples of Problems With Different Unit Types From the Textbook Analysis

Unit Type			Example
Continuous	Base-10 measure	metric (meter, liter, kilogram), currency, Celsius	*"There are 10.7 liters of water flowing into a bucket per minute. After 17.1 minutes, how many liters of water are in the bucket?"* *"Ben bought 4 sacks of flour. Each sack weighed 2.3 kg. How many kilograms of flour did Ben buy?"* *"Lou's temperature was 39.6C when he was sick. After he took medicine it dropped to 37.9C. How much did it drop?"*
	Nonbase-10 measure	imperial (inch, pound, gallon), time (seconds, minutes, hours), Fahrenheit	*"If a full 1 gallon jug of water is poured into a 1/2 gallon jug, how much water is left in the 1 gallon jug?"* *"Kari practiced the piano for 1/2 of an hour. Brandon practiced the piano for 1/3 of an hour. Who practiced longer?"* *"A steak weighed 2 1/2 lbs. After the fat was removed it weighed 2 1/4 lbs. What was the weight of the fat?"*
Countable	collective nouns (people, class of students), slices of a mass (pizza, pies, apples), discrete set (marbles, balloons, grapes, crayons)		*"Larry had 12 balloons. He popped 1/3 of them. How many balloons did Larry pop?"* *"If 7/12 of the nations present voted to send aid to flood victims, would the vote pass by a 2/3 majority?"* *"Keiko and Robert each got a pizza. Keiko's was cut into sixths. Robert's was cut into eighths. They ate half of their pizzas. How many more pieces did Robert eat?"*

problems, $n = 11$). Although currency is technically written in the form of a decimal, it has different properties than typical decimals (e.g., colloquially, we refer to $6.10 as "6 dollars and 10 cents," not "6.1 dollars"). We performed an additional chi-square test of independence to test the association between entity type and number type when currency problems were excluded, again finding that the two factors are significantly associated ($\chi^2(2, N = 762) = 49.79$, $p < .001$). Excluding currency problems, we found a similar alignment to entity type, with 70% of decimal problems using continuous entities and 57% of fraction problems using discrete entities.

Figure 3 shows the distribution of continuous base-10 ($n = 215$) and nonbase-10 problems ($n = 291$) that were represented by either decimals ($n = 288$) or by fractions ($n = 218$). Base-10 problems comprised 70% of the decimal problems, whereas nonbase-10 problems comprised 94% of the fraction problems. A chi-square test of independence between number type and unit type confirmed that there was a significant relationship between the two factors ($\chi^2(4, N = 506) = 354.8$, $p < .001$).

In summary, the textbook analysis revealed a pattern of alignment that is consistent with our entering hypotheses: Continuous entities were more likely to be represented with decimals than with fractions, whereas countable entities were more likely to be represented with fractions than with decimals. Also, as we predicted, the tendency to align continuous entities with decimals rather than with fractions was much more pronounced for entities measured with base-10 units (metric units and currency) than for nonbase-10 units (imperial units).

Experiment 1

The textbook analysis revealed that, by and large, math educators present their students with word problems in which decimals are paired with continuous entities and fractions are paired with countable entities, or with continuous entities measured in imperial

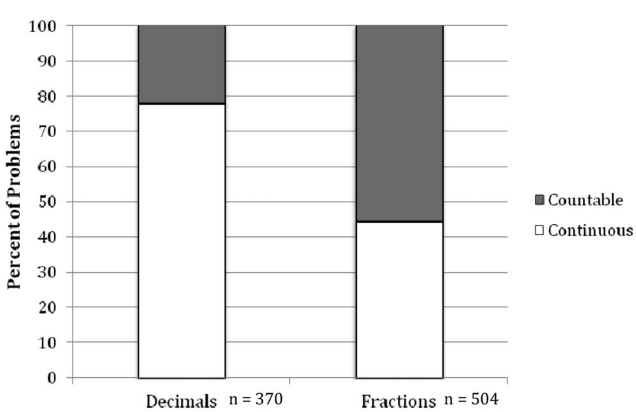

Figure 2. Percentage of decimal and fraction problems in the textbook analysis that were continuous or countable.

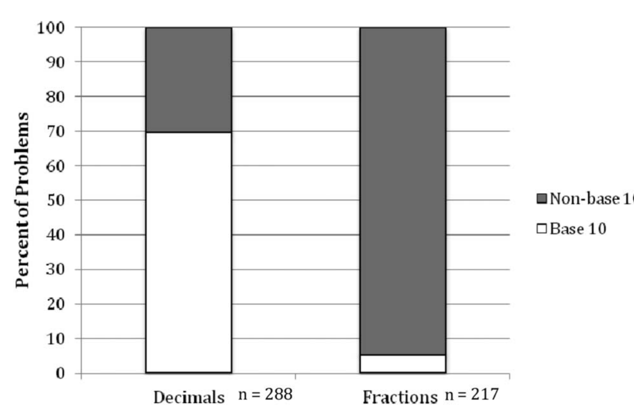

Figure 3. Percentage of continuous decimal and fraction problems in the textbook analysis that included either base-10 or nonbase-10 units.

units. Experiment 1 examined whether adult undergraduates, who were likely exposed to such problems in their early schooling, show the same pattern of alignment. To this end, we asked undergraduate students to generate word problems that contained either fractions or decimals, and examined the entities (countable vs. continuous) they described in their problems.

Method

Participants

A total of 130 undergraduates (males = 72; mean age = 19) from the University of Washington were included in the study, receiving course credit.[2] Half of these participants generated decimal problems and half generated fraction problems.

Materials and Design

The study had one factor: number type (fraction vs. decimal), which was manipulated between subjects.

Procedure

Participants completed the study using paper and pencil. They were given a single sheet of paper. At the top, they saw three examples of simple word problems with whole numbers, two involving countable object sets (30 marbles, 5 children), and one involving a mass entity (a 2-pound bag of sugar). All of the examples were presented with whole numbers, rather than either type of rational number, so as not to introduce any demand characteristics for the participants. Participants were then asked to generate two word problems with their own numbers. Depending on the condition, they were told that at least one of the numbers in their problems had to be a fraction (e.g., 1/4, 5/2), or that it had to be a decimal (e.g., 0.25, 1.3).

Results and Discussion

The constructed fraction and decimal problems were coded for continuity type and unit type using the same scheme that we had developed for the textbook analysis reported above. Examples of the problems generated, and the coding of these problems, are provided in Table 2. The results are summarized in Figures 4 and 5. As in the textbook analysis, decimal problems ($n = 130$) more often included continuous entities (72%). Conversely, fraction problems ($n = 130$) more often included countable entities (66%). A chi-square test confirmed that number type and continuity were significantly associated ($\chi^2(2, N = 260) = 42.0, p < .001$). As in the textbook analysis, we also conducted a chi-square test excluding currency problems (for decimals, $n = 31$; for fractions, $n = 8$), and still found a significant association between number type and entity type ($\chi^2(2, N = 221) = 33.47, p < .001$).

Figure 5 shows the distribution of continuous problems with base-10 and nonbase-10 units in the decimal and fraction problems. Overall, students generated more continuous problems with nonbase-10 units ($n = 90$) than with base-10 units ($n = 48$), perhaps reflecting a general preference of American students for imperial over metric units. Also, the instructions included an example with imperial units (pounds), but not one with metric units, which may have inadvertently primed participants to think of imperial rather than metric units. This preference was apparent in both fraction and decimal problems. Nonetheless, in accord with the hypothesized alignment of continuous unit type with number type, base-10 units were used more frequently with decimals (42%) than with fractions (21%), whereas nonbase-10 units were used more frequently with fractions (79%) than with decimals (58%). A chi-square test confirmed that, for continuous entities, unit type and number type were significantly associated ($\chi^2(4, N = 138) = 41.8, p < .001$).

The results of Experiment 1 closely match the pattern of results found in the textbook analysis. Much like the word problems constructed by math educators, college students generate word problems in which they tend to use decimals to represent continuous entities and fractions to represent discrete or countable entities. Also, for continuous entities, they are more likely to represent base-10 units with decimals than with fractions. Overall, these results indicate that, for both textbook writers and college students, a distinct pattern of alignment governs how rational numbers are used to represent particular types of entities.

Experiment 2

In both the textbook word problems and in the word problems generated by college students (Experiment 1) we found an association between rational number type and entity type. As we noted in the Introduction, the distinction between continuous and discrete entities is also reflected in the interpretation of diagrams and graphs (Bassok & Olseth, 1995; Shah et al., 1999). In Experiment 2 we investigated whether people preferentially associate fractions and decimals with different types of diagrammatic representations. We asked college students to choose either a continuous or a discrete depiction of fractions and decimals, which were paired with matched continuous or discrete entities. The goal of this study was to determine whether college students would associate fractions with discrete representations and decimals with continuous representations. Importantly, we tested whether this association interacts with the type of unit (metric or imperial) paired with the rational number. If the impact of unit type is primarily because of differences in ease of computation, then we might expect this variable to have less influence in Experiment 3, where the task does not require any sort of computation.

Method

Participants

The participants were 157 college students, 115 female, from the University of Washington. Participant ages ranged from 18–25 years (mean age of 19.4 years). All participants were enrolled in an introductory psychology course and received course credit for their participation.

[2] Originally a larger number of participants (91 in total) were included in the decimal condition in order to match the sample size with that of an alternative variant of the fraction condition, which was subsequently dropped from the design. The final sample of 65 was randomly selected from the set of 91 so as to equate sample sizes for the decimal and fraction conditions.

Table 2
Examples of Problems Generated in Experiment 1 With Different Unit Types

Unit Type			Example
Continuous	Base-10 measure	metric (meter, liter, kilogram), currency, Celsius	*"The cost of a candy bar is $1.25. If tax adds an additional $.10, how much is the candy bar?"* *"There is .7g of salt and 1.4g of sugar. What is the total weight of the two?"* *"If the radius of a cylinder is .5m and the height is 7.2 m, what is the volume?"*
	Nonbase-10 measure	imperial (inch, pound, gallon), time (seconds, minutes, hours), Fahrenheit	*"School is 6 1/2 miles away. If I drive 25 miles/hr to get there how long will it take me?"* *"If there is a bag of flour that is 1/4 full and then you add another bag that is also 1/4 full, how full is your bag after combining them?"* *"A 1 lb ground beef patty is combined with a 1/2 lb lump of pork. How much does the combination weigh?"*
Countable	collective nouns (people, class of students), slices of a mass (pizza, pies, apples), discrete set (marbles, balloons, grapes, crayons)		*"If you have a dozen eggs and your neighbor borrows 1/4 of them, how many are you left with?"* *"If there are 50 marbles in a full container, but 1/2 the container is gone, how many marbles are left?"* *"I cut a whole pizza into 1/4ths. If I eat 1/2 of the slices, how many slices are left?"*

Materials and Design

The study was a 2 (number type: fraction vs. decimal) × 2 (countable vs. continuous entity type) repeated-measures design. There were two trials of each type, for a total of eight trials per participant. Each participant saw eight different expressions, each including either a fraction or a decimal and either a countable (pens, sandwich, books, and banana) or continuous (kilometer, pound, mile, and kilogram) entity type. Four fractions were used (3/4, 5/8, 4/9, 2/7), and their equivalent decimals (.75, .63, .44, .29). For example, a participant might see "3/4 km" or ".75 sandwich." Assignments of entity type and number type were counterbalanced so that half of the participants received a fraction with a particular entity (e.g., 3/4 sandwich) and half received the equivalent decimal with that same entity (e.g., .75 sandwich). Thus, each participant saw eight of the 16 possible pairings of number and entity type.

The dependent measure was whether participants selected a continuous circle representation or a discrete dot representation for the number type-entity type expressions (see Figure 6). Critically, the representation options were the same for all of the statements. Both of the representations depicted the value of 1/2 (.50), which was not used in any of the fractions or decimals given in the statements. The choice of representation type thus could only be guided by its abstract form (continuous or discrete), rather than by matches of specific values.

Procedure

Participants were given eight expressions that paired number type and entity type, and shown the two different diagrammatic representations depicted in Figure 6. For each expression participants were instructed to choose which type of diagram (circle or dots) they would prefer to use to represent it.

Results and Discussion

Figure 7 depicts the proportion of total times the continuous representation versus discrete representation was chosen for a

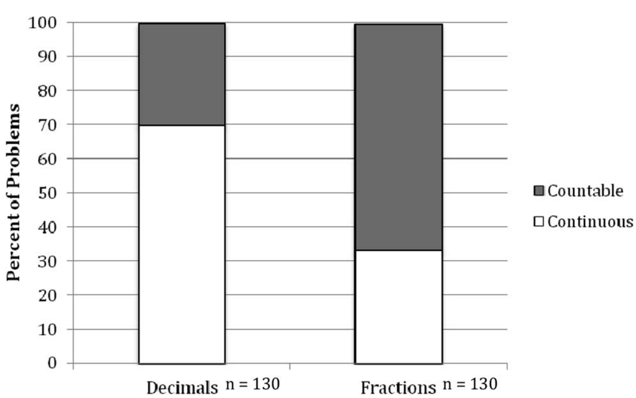

Figure 4. Percentage of decimal and fraction problems in Experiment 1 that were continuous or countable.

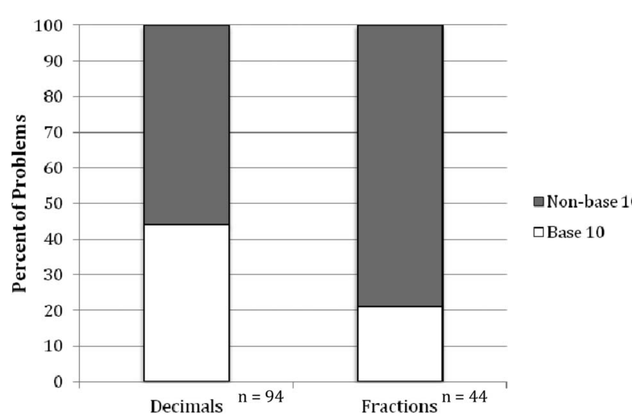

Figure 5. Percent of continuous decimal and fraction problems in Experiment 1 that included base-10 or nonbase-10 units.

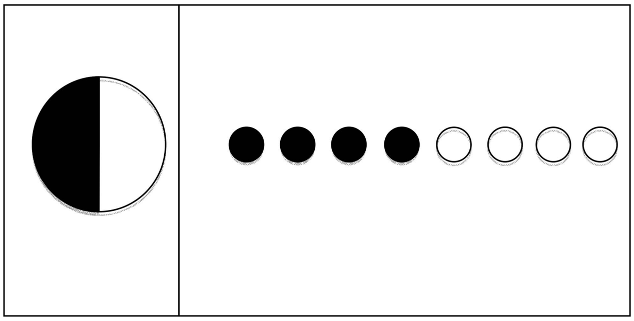

Figure 6. Options provided to represent continuous (circle) and discrete (dots) representations in Experiment 2.

given combination of entity type and number type. Collapsing across entity type, for decimal expressions participants selected the continuous representation 65% of the time; whereas for fraction expressions participants chose the discrete representation 59% of the time. Because each participant received just two items of each type, we used a nonparametric sign test to evaluate differences in selecting the continuous versus discrete display for decimals versus fractions. Of the 157 participants, 85 selected the continuous display more often for decimal than fraction trials, 38 selected the continuous display more often for fraction than decimal trials, and 35 showed no preference. A sign test revealed that more participants showed an overall preference for continuous displays with decimals versus fractions ($Z = -4.15$, $p < .001$).

Collapsing across notation type, participants chose the continuous circle display 60% of the time when given continuous entities, but chose the discrete dot display 55% of the time when given the discrete entities. We coded whether each participant selected the continuous display more for continuous than discrete entities, or vice versa. Of the 157 participants, 82 selected the continuous display more often for continuous than discrete entities, 32 selected the continuous display more often for discrete than continuous entities, and 44 showed no preference. A sign test revealed that there was a significant difference in the preference for continuous displays with continuous entities versus discrete entities ($Z = -4.59$, $p < .001$).

It is interesting to note that, although we paired fractions and decimals with both metric and imperial units, unit type of the continuous entities did not affect participants' choices of the continuous versus discrete diagrammatic representations. For continuous problems with decimals, the continuous representation was selected 68% of the time when the units were base 10 and 73% of the time when the units were nonbase 10. For continuous problems with fractions, the continuous representation was selected 50% of the time when the units were base 10 and 50% of the time when the units were nonbase 10. This lack of a unit-type effect indicates that this variable does not, in itself, affect the perceived continuity versus discreteness of the situation model. Rather, the effects of unit type we found in the previous studies (textbook analysis and Experiment 1) appear to reflect the relative ease of representing continuous magnitudes with either decimals or with fractions.

The results of Experiment 2 show that, in addition to the direct impact of entity type on the selection of a continuous versus discrete representation, the participants in Experiment 2 preferred to represent decimals with a continuous diagram but preferred to represent fractions with a discrete diagram. Thus, continuous entities paired with decimals (e.g., .44 km) showed the strongest preference for the continuous representation, whereas countable entities paired with fractions (e.g., 4/9 pens) showed the strongest preference for the discrete representation. These results provide strong support for the alignment between the perceived continuity or discreteness of rational numbers (decimals or fractions, respectively) and the continuity or discreteness of the modeled entities.

General Discussion

Results of the textbook analysis and of two experiments with college students are consistent with our entering analysis of alignment between the format of rational numbers and the entity type these numbers could meaningfully represent. Although the hypothesized alignment was not absolute, decimals were typically used to represent continuous entities, whereas fractions were more likely to represent discrete than continuous entities. In the word problems generated by textbook writers and by college students (Experiment 1), we also found a strong correspondence between unit type of continuous entities (base-10 vs. nonbase-10) and the format of rational numbers (decimals vs. fractions). However, unit type had no effect on participants' choices of continuous versus discrete diagrammatic representations (Experiment 2), a task that does not require mathematical computation. The effect of unit type thus appears to reflect the relative ease of representing continuous

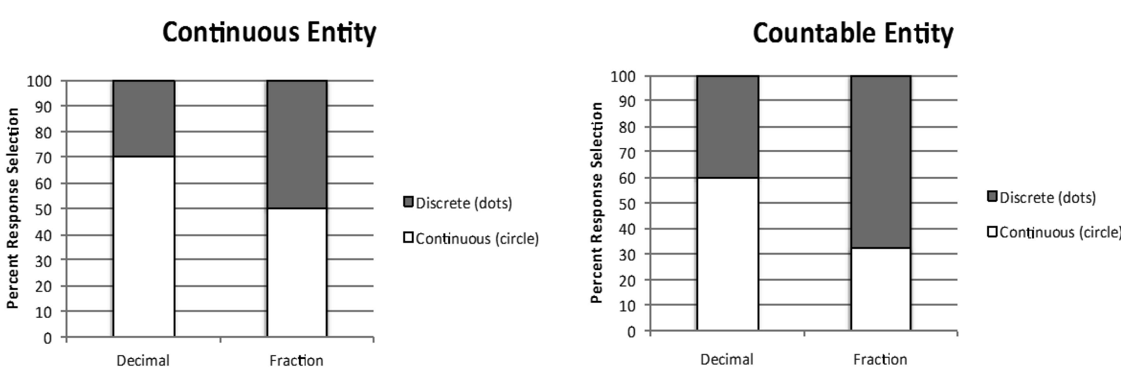

Figure 7. Percentage response selection by number type for trials with continuous entities (A) and countable entities (B) in Experiment 2.

magnitudes with either decimals or with fractions. In contrast, the preferential alignment of fractions with discrete representations and decimals with continuous representations has a conceptual basis, and hence is found even in a task that does not require computation.

The results of Experiment 2 also showed that people view aligned number representations of entity type (e.g., 4/9 pens or 0.44 km) as better exemplars of discreteness or continuity than misaligned number representations of the same entities (e.g., 0.44 pens or 4/9 km). This pattern of alignments suggests that people view the symbolic notations of rational numbers as either discrete (fractions) or continuous (decimals). Although this hypothesis has yet to be tested directly (e.g., by asking people to select continuous or discrete representations for numbers without specifying their units), it is consistent with recent work showing the impact of notational tools on mathematical reasoning (Braithwaite & Goldstone, 2013; Fisher, Borchert, & Bassok, 2011; Landy & Goldstone, 2007; Zahner & Corter, 2010).

We motivated the hypotheses tested in the present paper by the framework of semantic alignment, which postulates that conceptual distinctions such as continuity versus discreteness can guide people's mappings of mathematical expressions onto situations. However, an alternative interpretation is that the performance of college students, and the correspondence between their performance and the textbook examples, merely reflects the students' early exposure to this alignment in the textbook examples. Of course, this account would have to explain why textbook writers chose such examples. To the extent that they have attempted, consciously or unconsciously, to find the best real-life examples that correspond to the target mathematical concepts, our results may reflect a cognitively natural alignment between discrete versus continuous entities and their mathematical representations with fractions versus decimals. The fact that the alignments we have identified may have a basis in the mathematical nature of fractions and decimals (see Footnote 1), and hence may be nonarbitrary, lends further credence to the semantic-alignment hypothesis. Nonetheless, further research will be required to resolve the "chicken and egg" dilemma concerning the basic origin of these alignments.

As we have pointed out in the introduction, continuity versus discreteness is a basic ontological distinction that affects children's understanding of integers through counting of discrete entities, and (later on) through measurement of continuous entities that have been parsed into discrete units (e.g., Mix et al., 2002a, Mix, Huttenlocher, & Levine, 2002b; Gelman, 1993; Nunes et al., 1993; Gelman, 2006; Rips et al., 2008). The distinction between continuity and discreteness is preserved throughout the mathematical curriculum. As in the initial cases of counting and measurement, discrete concepts (at least in the typical curricula employed in the United States) are always taught before their continuous counterparts (e.g., first arithmetic progressions, then linear functions). Consistent with this typical instructional progression, students learn fractions (kindergarten through 3rd grade) before they are introduced to decimals (3rd grade). Although mathematics educators do not make an explicit claim that the transition from fractions to decimals corresponds to the transition from countable to continuous entities, our findings strongly suggest that this is the case.

Applications to Instruction

One important application of the present findings concerns how using fractions and decimals to model discrete and continuous entities may affect reasoning about such entities. The two formats of rational numbers, together with their respective alignments to discrete and continuous entities, are differentially suited for different reasoning tasks. In a recent study, DeWolf, Bassok, and Holyoak (in press) found that fractions allow people to better represent bipartite relations between discrete sets than do decimals. This difference arises because fractions maintain the mapping of distinct countable sets onto the numerator and the denominator, whereas decimals obscure this mapping. At the same time, decimals afford direct mapping onto a mental number line and, therefore, allow for easier magnitude assessment than do fractions (DeWolf, Grounds, Bassok, & Holyoak, 2014; Iuculano & Butterworth, 2011).

These recent findings, together with the results of the current study, suggest that it will be useful for educators to be aware of these alignments when developing word problems or questions in which rational numbers are used to model proportions of entities. While we find evidence that such alignments are already reflected in textbooks, this is not done in a way that explicitly highlights the connection between the types of entities and different formats for rational numbers. Making this connection clearer to students may help them to interpret the goals of modeling such entities, and clarify how different formats can be used and manipulated to suit the specific goals of the modeling task. For example, modeling complex relationships between countable sets may be better carried out with fractions, whereas expressing a measurement from a ruler may be better suited for decimal notation.

The present findings are also interesting in light of recent research on the understanding of magnitudes of rational numbers by both children and adults. A popular test of knowledge of the magnitudes of rational numbers is a number line estimation task, in which a participant places a fraction on a continuous number line, usually ranging from 0 to 1 (Siegler, Thompson, & Schneider, 2011). Both adults and children are more accurate when performing this task with decimals rather than fractions (Iuculano & Butterworth, 2011). However, Siegler and his colleagues have shown that ability to perform well on this task with fractions is highly predictive of later performance in mathematics (Jordan et al., 2013; Siegler et al., 2012). Siegler and his colleagues have argued that asking students to place fractions on a continuous number line is one of the single best ways to improve students' understanding of fractions (Siegler et al., 2011; Siegler, Fazio, Bailey, & Zhou, 2013).

The number line estimation task requires mapping a fraction onto a continuous entity, which our results suggest would be a difficult operation. It may be that the process of taking a continuous representation, such as a number line, and parsing it into meaningful pieces for the purposes of alignment to a fraction, can help children gain a better understanding of both the magnitude of the fraction and the relationship between its numerator and denominator. Therefore, it is not necessarily the case that each type of rational number should only be used with a specific type of entity.

More generally, understanding of the natural alignment between entity type and rational numbers, and capitalizing on it, may be useful in teaching rational numbers. Given that we know students are particularly prone to misconceptions with rational numbers (Stafylidou & Vosniadou, 2004; Ni & Zhou, 2005; Stigler, Givvin, & Thompson, 2010), making use of this natural alignment may help students to use their knowledge of entities in the real world to bootstrap their knowledge of rational numbers. Interestingly, despite the prevalence of this alignment in textbooks across many grade levels, textbooks never actually address it explicitly. The alignment seems to be implicit, and is not explicitly taught even for adults. Teaching with this alignment in mind, and even explicitly using it, may provide a useful stepping stone for children learning natural numbers. In addition, having students engage in tasks in which they need to actively parse a continuous representation, or conversely sum over a discrete representation to align it with a decimal value, may provide a useful tool for bolstering understanding of the relation between the representations of entities and the rational numbers themselves.

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