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Semantic Alignment of Fractions and Decimals with Discrete Versus Continuous Entities: A Textbook Analysis

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Abstract

When people use mathematics to model real-life situations, their modeling is often mediated by *semantic alignment* (Bassok, Chase, & Martin, 1998): The entities in a problem situation evoke semantic relations (e.g., tulips and vases evoke the functionally asymmetric "contain" relation), which people align with analogous mathematical relations (e.g., the non-commutative division operation, tulips/vases). Here, we applied the semantic-alignment framework to understand how people use rational numbers as models of discrete and continuous entities. A textbook analysis revealed that mathematics educators tend to align the discreteness vs. continuity of the entities in word problems (e.g., marbles vs. distance) with distinct symbolic representations of rational numbers—fractions vs. decimals, respectively. We discuss the importance of the ontological distinction between continuous and discrete entities to mathematical cognition, the role of symbolic notations, and possible implications of our findings for the teaching of rational numbers.

Keywords: Number concepts, continuous and discrete quantities, fractions, decimals, semantic alignment

Introduction

Word problems are a classic tool that mathematics educators use to help children understand abstract mathematical concepts and the appropriate conditions of using them when solving real-life problems. Word problems describe simple situations involving various entities that can be modeled by the target mathematical concepts. For example, the mathematical concept of a fraction is often illustrated with a word problem describing a pizza that is shared by several children. The pizza is sliced into n equal slices, and each slice is denoted by the fraction $1/n$. Importantly, in order to be effective as examples of the target mathematical concepts, the situations described in the word problems, or "situation models", have to be analogous to their mathematical representations, or "mathematical models" (Kintsch & Greeno, 1985). For example, in the above pizza problem, the mathematical concept of a fraction requires that the pizza slices be equal in size.

Semantic Alignment and Mathematical Problem Solving

People who have extensive experience with solving word problems are highly systematic in selecting mathematical models that correspond to the situation models (e.g., Bassok, Chase, & Martin, 1998; Bassok, Wu, & Olseth, 1995; Dixon, 2005; Dixon, Deets & Bangert, 2001; Mochon & Sloman, 2004; Sherin, 2001; Waldmann, 2007). But how do students and mathematics educators decide that particular situations are analogous to particular mathematical models? Bassok et al. (1998) have proposed that such modeling decisions are guided by *semantic alignment*. The essence of the semantic-alignment process is that the entities in a problem situation elicit particular semantic relations between them (e.g., tulips and vases are likely to evoke the functionally asymmetric "contain" relation), which people then align with structurally analogous mathematical relations (e.g., the non-commutative division operation, tulips/vases). Both children and adults find it easier and more natural to solve or construct semantically-aligned rather than misaligned word problems (e.g., tulips/vases rather than tulips/roses; Martin & Bassok, 2005). For many adults the process of semantic alignment is highly automatic (Bassok, Pedigo, & Oskarsson, 2008; Fisher, Bassok, & Osterhout, 2010).

In addition to semantic alignments of inferred object relations, the entities in word problems lead people to infer the continuity vs. discreteness of situation models and of the corresponding mathematical models. To illustrate, a word problem that describes constant change in the value of a coin evokes a situation model of continuous change, whereas a word problem that describes constant change in salary evokes a situation model of discrete changes. These distinct situation models lead college students to select corresponding continuous or discrete visual representations of constant change, to produce corresponding continuous "swipe" or discrete "tap" gestures when describing the problems, and to generate qualitatively distinct mathematical solutions (e.g., average vs. repeated addition) to otherwise isomorphic word problems (Alibali et al., 1999; Bassok & Olseth, 1995).

Semantic alignments of discreteness and continuity are consistent with well-established linguistic analyses and research on the development of mathematical cognition. Specifically, the linguistic distinction between count and mass nouns (e.g., marbles vs. water, respectively; see Bloom, 1994; Bloom & Wynn, 1997) demonstrates a fundamental ontological distinction that affects how people parse the world. For example, Spelke, Brelinger, Macomber and Jacobson (1992) have argued that young babies use this distinction to discriminate between objects: Continuity of motion indicates that a single object is moving in space, whereas discontinuity indicates the existence of more than one object. Importantly, this distinction plays a crucial role in the development of "number sense" (Dehaene, 1997). According to Dahaene and his colleagues, an approximate sense of magnitude for mass entities is evolutionarily more primitive than exact calculations with discrete objects (Feigenson, Dehaene & Spelke, 2004; Dehaene, 1997).

Counting, or "enumerating" (Gelman & Gallistel, 1978), is the first opportunity for children to explicitly align entities with numbers. The counting process involves one-to-one mapping of consecutive integers to distinct objects (e.g., stickers, chairs), such that a child increments the integer magnitude simultaneously with the act of moving through the set of objects, with the last number denoting the set cardinality (e.g., 3 stickers, 4 chairs). The use of integers in counting discrete objects precedes their use in exact quantification of continuous entities (e.g., 2 lbs of sugar, 3 feet), which requires explicitly parsing continuous entities into countable measurement units (Miller, 1984; Mix, Huttenlocher, & Levine, 2002a; Nunes, Light, & Mason, 1993).

Alignment of Discrete and Continuous Entities with Fractions and Decimals

Whereas prior research has documented semantic alignments between situation models and mathematical models of word problems, the present study aimed to examine whether people treat numbers as mathematical models of quantities. Specifically, we examined whether people use different symbolic notations of rational numbers —fractions and decimals—to represent parts of discrete (or countable) and continuous entities (e.g., ½ of the marbles, 0.5 liter of water, respectively).

Figure 1 depicts the hypothesized alignments. Fractions have a bi-partite structure (a/b), which expresses the value of the part (the numerator *a*) and the whole (the denominator *b*). Decimals represent the one-dimensional magnitude of fractions (*a/b* = *c*) expressed in the standard base-10 metric system. The fraction format is well suited for representing sets and subsets of discrete entities (e.g., balls, children) that can be counted and aligned with the values of the numerator (*a*) and the denominator (*b*) (e.g., 3/8 of the balls are red). By contrast, the magnitude of fractions (*c*) is a one-dimensional decimal representation. The decimal format fails to capture the relation between a subset and a set and may imply partition of non-divisible entities (0.375 of the balls are red). Thus, whereas discrete entities can be readily aligned with fractions, they are poorly aligned with decimals. Note that, as is the case with integer representations, the fraction format can be readily used to represent continuous entities that were discretized—parsed into distinct equal-size units—and therefore counted (e.g., 5/8 of a pizza). Decimal representations of such discretized continuous entities are meaningful, but appear to be less intuitive than fractions (e.g., 0.625 of a pizza).

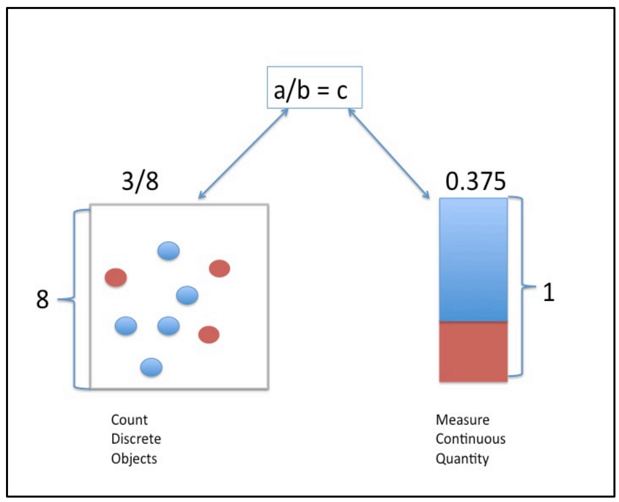

Figure 1. Alignment of discrete and continuous entities with fractions and decimals.

Whereas fractions seem to be better aligned with sets of discrete (or discretized) entities than are decimals, the one-dimensional format of decimals seems to be best suited to model one-dimensional magnitudes of continuous entities. This alignment should be especially strong when decimals (base-10) are used to model entities that have corresponding metric units (0.375 meters, 0.72 liters, $0.33). When continuous entities have non-metric units (e.g., imperial measures, or measures of time), their magnitude can be meaningfully represented by both decimals (0.67 ft) and fractions (2/3 of a foot). This is the case because, as we have mentioned earlier, fractions can be adapted to any unit base and therefore represent continuous entities that are parsed into countable units (e.g., inches, minutes).

The above analysis suggests that semantic knowledge about the discreteness or continuity of the entities in word problems will lead people to select either fractions or decimals (respectively) as symbolic mathematical models of these entities. Here, we report results from a textbook analysis that examined whether math educators who, one would assume, aim to help students understand rational numbers, choose problems that align fractions with discrete entities and align decimals with continuous entities.

Method

Design and Materials

We examined the *Addison-Wesley Mathematics* (1989) textbook series from grades kindergarten through 8th grade. This particular textbook series was chosen because it is representative of the mathematics teaching that most college undergraduates who participated in a follow-up study (conducted between 2010 and 2012, not reported here) would have received in their early education (i.e., during the 1990s). The K-8 grade levels were selected because they cover the main introduction and use of rational numbers in math curricula prior to the start of formal algebra. We analyzed all the problems that involved rational numbers, a total of 874 problems (504 with fractions, 370 with decimals).

Problem Coding

We developed a coding scheme that categorized problems by their number type (*fraction* vs. *decimal*) and entity type (*continuous* vs. *countable*). Problems were classified as *fraction* or *decimal* based on the number type that appeared in the problem text or were called for in the answer. (None of the problems included both fractions and decimals.) Problems were classified as *continuous* or *countable* based on the entities in the problems. Continuous problems involved entities that are referred to linguistically as "mass nouns" (e.g., those varying continuously in weight, volume, or length). Importantly, these continuous entities were treated as wholes (e.g., the length of a string) and were not explicitly broken down into smaller countable pieces (as in a string that was cut into three equal parts). We also coded the unit type used in the continuous problems (base-10, yes or no) in order to assess whether the base-10 format of decimals is used more often with readily-aligned base-10 units than with non-base-10 units.

Countable problems involved either discrete or explicitly discretized entities. Discrete entities were sets of individual objects that cannot be broken down into natural equal units (e.g., marbles, balloons, or grapes). Continuous entities that were parsed into equal countable parts (e.g., an apple cut into slices, or a rectangle divided into equal squares) were coded as discretized. In addition, the discretized category encompassed collective nouns (e.g., people, class), which are collections of countable objects (a person, a student). Examples of the coded problems appear in Table 1.

One research assistant coded all the problems using the above coding scheme. In order to assess inter-rater reliability, a second researcher coded a random sample of 350 problems (i.e., 40% of the total problems). The second coder was blind to the original coder's judgments. The two coders agreed on 336 (96%) of the sampled problems. A third researcher, who was blind to the first two coders' judgments, then coded the 14 problems on which the first two coders had differed. These problems were then placed into whichever category it was assigned by two of the three coders.

Results

Figure 2 presents the distribution of the textbook problems. Of the 874 total problems, 504 used fractions and 370 used decimals. Continuous entities comprised a large majority of the decimal problems (78%). In a complementary way, countable entities comprised a majority of the fractions problems (57%). A chi-square test of independence between number type and continuity

Table 1: Examples of problems with different unit types from the textbook analysis

Entity Type	Unit Type	Example
Continuous	Base-10 measure: metric (meter, liter, kilogram), currency, Celsius	"There are 10.7 liters of water flowing into a bucket per minute. After 17.1 minutes, how many liters of water are in the bucket?" "Ben bought 4 sacks of flour. Each sack weighed 2.3 kg. How many kilograms of flour did Ben buy?"
	Non-base-10 measure: imperial (inch, pound, gallon), time (seconds, minutes, hours), Fahrenheit	"If a full 1 gallon jug of water is poured into a 1/2 gallon jug, how much water is left in the 1 gallon jug?" "A steak weighed 2 1/2 lbs. After the fat was removed it weighed 2 1/4 lbs. What was the weight of the fat?"
Countable	collective nouns (people, class of students), slices of a mass (pizza, pies, apples), discrete set (marbles, balloons, grapes, crayons)	"Larry had 12 balloons. He popped 1/3 of them. How many balloons did Larry pop?" "If 7/12 of the nations present voted to send aid to flood victims, would the vote pass by a 2/3 majority?" "Keiko and Robert each got a pizza. Keiko's was cut into sixths. Robert's was cut into eighths. They ate half of their pizzas. How many more pieces did Robert eat?"

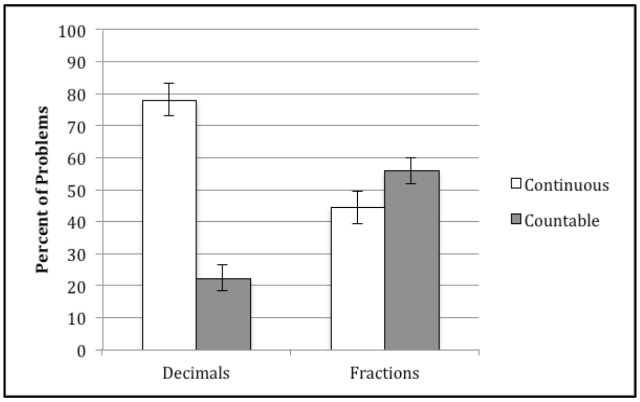

Figure 2. Percentage of decimal and fraction problems in the textbook analysis that were continuous or countable.

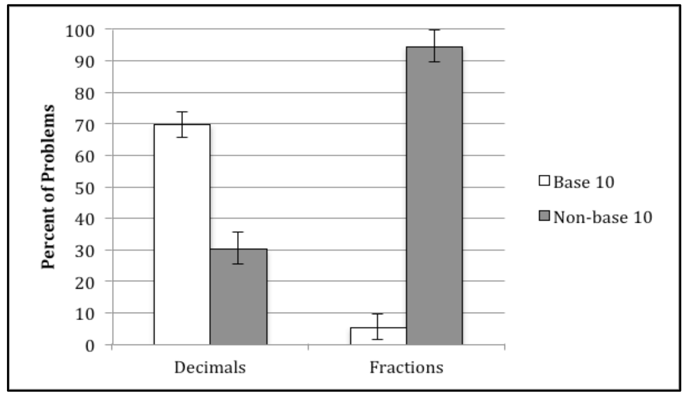

Figure 3. Percentage of continuous decimal and fraction problems in the textbook analysis that included either base-10 or non-base-10 units.

confirmed that the two factors are significantly associated (χ^2 (2, N = 874) = 115.7, p < .001).

Figure 3 shows the distribution of continuous base-10 (n = 215) and non-base-10 problems (n = 291) that were represented by either decimals (n = 289) or by fractions (n = 217). Base-10 problems comprised 70% of the decimal problems, whereas the non-base-10 problems comprised 94% of the fraction problems. A chi-square test of independence between number type and unit type confirmed that there was a significant relationship between the two factors (χ^2(4, N = 874) = 354.8, p < .001).

In summary, the textbook analysis revealed a pattern of alignment that is consistent with our entering hypotheses: continuous entities were more likely to be represented with decimals than with fractions, whereas countable entities were more likely to be represented with fractions than decimals. Also, as we predicted, the tendency to align continuous entities with decimals rather than with fractions was much more pronounced for entities with base-10 units (metric unit and currency) than for non-base-10 units (imperial units). Together, these findings reveal an alignment between decimals and continuous entities for the base-10 scale, and an alignment of fractions with countable entities for more idiosyncratic scales.

Discussion

The results of the textbook analysis are consistent with our entering analysis of alignment between the format of rational numbers and the entity type these numbers could meaningfully represent. Although the hypothesized alignment was not absolute, decimals were typically used to represent continuous entities, whereas fractions were more likely to represent discrete than continuous entities. This suggests that fractions and decimals have different implications for the types of entities that we expect to see them paired with. In the word problems generated by textbook writers, we also found a strong correspondence between unit type of continuous entities (base-10 vs. non-base-10) and the format of rational numbers (decimals vs. fractions).

As we pointed out in the Introduction, continuity versus discreteness is a basic ontological distinction that affects children's understanding of integers through counting of discrete entities, and (later on) through measurement of continuous entities that have been parsed into discrete units (e.g., Mix et al., 2002a, 2002b; Gelman, 1993; Nunes et al., 1993; Gelman, 2006; Rips, Bloomfield & Asmuth, 2008). The distinction between continuity and discreteness is preserved throughout the mathematical curriculum. As in the initial cases of counting and measurement, discrete concepts are always taught before their continuous counterparts (e.g., first arithmetic progressions, then linear functions). Consistent with this typical instructional progression, students learn fractions (K through 3rd grade) before they are introduced to decimals (3rd grade). Although mathematics educators do not make an explicit claim that the transition from fractions to decimals corresponds to the transition from countable to continuous entities, our findings strongly suggest that this is indeed the case.

The two symbolic notations of rational numbers, together with their respective alignments to discrete and continuous entities, are differentially suited for different reasoning tasks. In a recent study, DeWolf, Bassok, and Holyoak (2013) found that fractions allow people to better represent bipartite relations between discrete sets than do decimals. This difference arises because fractions maintain the mapping of distinct countable sets onto the numerator and the denominator, whereas decimals obscure this mapping. At the same time, decimals afford direct mapping onto a mental number line and, therefore, allow for easier magnitude assessment than do fractions (DeWolf, Grounds, Bassok, & Holyoak, 2014; Iuculano & Butterworth, 2011).

The present findings are interesting in light of recent research on the understanding of magnitudes of rational numbers by both children and adults. A popular test of knowledge of the magnitudes of rational numbers is a number-line estimation task, in which a participant places a fraction on a continuous number line, usually ranging from 0 to 1 (Siegler et al., 2011). Both adults and children are more accurate when performing this task with decimals rather than fractions (Iuculano & Butterworth, 2011). However, Siegler and his colleagues have shown that ability to perform well on this task with fractions is highly predictive of later performance in mathematics (Jordan et al., 2013; Siegler et al., 2013). The number-line estimation task requires mapping a fraction onto a continuous entity, which our results suggest would be a difficult operation. It may be that the process of taking a continuous representation, such as a number line, and parsing it into meaningful pieces for the purposes of alignment to a fraction, can help children gain a better understanding of both the magnitude of the fraction and the relationship between its numerator and denominator.

The present textbook analysis suggests that there is a correspondence between how educators model discrete and continuous quantities with fractions and decimals. In a follow-up study (Rapp, Bassok, DeWolf & Holyoak, under review), we examined whether college students would spontaneously honor this alignment when prompted to create or solve word problems involving rational numbers. The results revealed a pattern qualitatively similar to that observed in the textbook analysis. The performance of college students, and the correspondence between their performance and the textbook examples, might simply indicate that students' selective use of fractions and decimals as models of discrete or continuous entities reflects their early exposure to this alignment in the textbook examples. Of course, this account would have to explain why textbook writers chose such examples. To the extent that math educators have attempted, consciously or unconsciously, to find the best real-life examples that correspond to the target mathematical concepts, the observed pattern of alignment may instead reflect a cognitively natural correspondence between discrete versus continuous entities and their mathematical representations with fractions versus decimals.

More generally, understanding of the natural alignment between entity type and rational numbers, and capitalizing on it, may be useful in teaching rational numbers. Given that we know students are particularly prone to misconceptions with rational numbers (Staflyidou & Vosniadou, 2004; Ni & Zhou, 2005; Stigler et al., 2010), making use of this natural alignment may help students to use their knowledge of entities in the real world to bootstrap their knowledge of rational numbers. Interestingly, despite the prevalence of this alignment in textbooks across many grade levels, textbooks never actually address it explicitly. The alignment seems to be implicit, and is not explicitly taught even for adults. Teaching with this alignment in mind, and even explicitly using it, may provide a useful stepping-stone for children learning natural numbers. In addition, having students engage in tasks in which they need to actively parse a continuous representation, or conversely, sum over a discrete representation to align it with a decimal value, may provide a useful tool for bolstering understanding of the relation between the representations of entities and the rational numbers themselves.

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