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Brief Report

From rational numbers to algebra: Separable contributions of decimal magnitude and relational understanding of fractions

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ABSTRACT

To understand the development of mathematical cognition and to improve instructional practices, it is critical to identify early predictors of difficulty in learning complex mathematical topics such as algebra. Recent work has shown that performance with fractions on a number line estimation task predicts algebra performance, whereas performance with whole numbers on similar estimation tasks does not. We sought to distinguish more specific precursors to algebra by measuring multiple aspects of knowledge about rational numbers. Because fractions are the first numbers that are relational expressions to which students are exposed, we investigated how understanding the relational bipartite format (a/b) of fractions might connect to later algebra performance. We presented middle school students with a battery of tests designed to measure relational understanding of fractions, procedural knowledge of fractions, and placement of fractions, decimals, and whole numbers onto number lines as well as algebra performance. Multiple regression analyses revealed that the best predictors of algebra performance were measures of relational fraction knowledge and ability to place decimals (not fractions or whole numbers) onto number lines. These findings suggest that at least two specific components of knowledge about rational numbers—relational understanding (best captured by fractions) and grasp of unidimensional magnitude (best captured by decimals)—can be linked to early success with algebraic expressions.

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Introduction

Given the well-documented difficulties that American students often experience in learning algebra and more advanced topics in mathematics (Gonzales et al., 2008; Richland, Stigler, & Holyoak, 2012; Smith & Thompson, 2007), it is important to identify those aspects of earlier mathematics that predict success or failure on advanced topics. Decomposing the prerequisites for success at algebra can potentially guide theoretical analyses of the mental representation of mathematics and also aid in developing more effective instructional strategies. Recent work suggests that knowledge of rational numbers, notably fractions, is closely linked to later success in mathematics. For example, in a large sample of students from the United States and the United Kingdom, Siegler and colleagues (2012) found that fraction knowledge, measured by basic arithmetic and conceptual questions, predicted algebra knowledge and math achievement in general at 16 years of age (beyond what could be predicted from knowledge of whole numbers). Other researchers have also found significant connections between fraction knowledge (including basic conceptual knowledge and performance on tasks that assess grasp of fraction magnitude) and algebra performance (Booth & Newton, 2012; Booth, Newton, & Twiss-Garrity, 2014; Brown & Quinn, 2007; Empson & Levi, 2011).

Although there seems to be an important link between fraction understanding and algebra performance, the nature of this link has yet to be firmly established. Wu (2009) and Siegler, Thompson, and Schneider (2011) emphasized the fact that a fraction, like any other type of number, can be placed on a number line. This understanding requires an integration of procedural and conceptual knowledge about fractions and magnitudes. Two recent studies by Booth and colleagues (Booth & Newton, 2012; Booth et al., 2014) have provided support for this hypothesis. Among middle school students taking elementary algebra classes, a significant correlation was observed between performance on a task requiring estimates of the positions of fractions on a number line and a subsequent algebra test that included problems requiring solving problems, knowledge of critical features in algebraic equations, and coding of equations. Number line estimation with fractions was a stronger predictor of algebra performance than declarative fraction knowledge, a measure of procedural knowledge of how to use fractions in equations, or number line estimation with whole numbers. These findings raise the possibility that a key link between knowledge of rational numbers and algebra performance may involve understanding of fraction magnitudes.

Although understanding of magnitudes is without question a core aspect of mathematical knowledge, there are good reasons to believe that understanding of mathematical *relations* is also critical in grasping algebra. For example, in the algebraic expression $x = 4y$, the value of the variable x is expressed in relation to that of y without any specific magnitude being assigned to either. In recent work (DeWolf, Bassok, & Holyoak, 2015; Rapp, Bassok, DeWolf, & Holyoak, in press), we have emphasized that fractions, with their bipartite a/b structure, naturally convey the relations between the numerator and the denominator (typically two countable sets). Of course, a fraction also represents the magnitude that corresponds to the division of a by b. This duality in the roles of fractions as mathematical representations of relations and magnitudes is similar to the duality of algebraic expressions (Sfard & Linchevski, 1994). Students must understand that they can use algebraic expressions to represent both the relations between quantities and the process used to find an unknown quantity. For example, the quantity of four boxes of equal weight can be represented as $4w$ without knowing the actual magnitude of a box's weight. The expression $4w$ represents the combined weight of the boxes and the process (multiplication) that could be used to determine the total weight given the actual weight of one box. Thus, students' conceptual understanding of fractions as representing both relations and magnitudes may be an important precursor for their subsequent understanding of algebraic expressions.

Interestingly, whereas fractions represent both relations and numerical magnitudes, magnitude equivalent decimals lose the relational structure inherent in a fraction and more directly express one-dimensional magnitude. Studies have shown that magnitude comparisons can be made much more quickly and accurately with decimals than with fractions (DeWolf, Grounds, Bassok, & Holyoak, 2014; Iuculano & Butterworth, 2011) but that fractions are more effective than decimals in tasks such as relation identification and analogical reasoning, for which relational information is paramount (DeWolf et al., 2015). Because fractions are the first numbers with an internal relational structure that students are taught, understanding of fractions as relations may be a key predictor of early algebra success. Insofar as understanding magnitudes is also important for grasping algebra, magnitude tasks involving decimals (which express magnitudes more directly than fractions) may be more predictive than magnitude tasks based on fractions.

Currently, no study has assessed tasks involving fractions as well as tasks involving decimals as predictors of algebra performance, nor has any study attempted to tease apart relational and magnitude knowledge as predictors. Hence, our aim in the current study was to better distinguish among possible links between specific types of rational number knowledge and early algebra performance. We extended the general design of the studies by Booth and Newton (2012) and Booth et al. (2014). In addition to assessing magnitude knowledge using number line tasks with whole numbers and fractions, we included a similar task with decimals. By comparing the predictive power of magnitude tasks with fractions and with decimals, we sought to determine whether fraction magnitude is a uniquely important predictor of algebra performance or whether knowledge of rational number magnitude in general (perhaps better assessed using decimals than fractions) is what is critical.

In addition, we added a measure designed to test students' understanding of fractions as relations. Importantly, performing well on this task did not require calculation of any particular magnitude. If knowledge of fractions predicts algebra performance because of transfer based on the status of fractions as relational expressions, then the relational test may provide a novel and unique predictor of success in algebra.

Method

Participants

All students were enrolled in introductory pre-algebra courses from two suburban Los Angeles schools. A total of 65 seventh-grade students (mean age = 12.4 years, 26 male and 39 female) participated in the study near the end of the school year. Students were from five different classes consisting of students with a substantial range of skill levels.

Measures and materials

Number line estimation tasks

To measure magnitude knowledge, we adopted a pencil-and-paper number line estimation task that has been used in many previous studies, including that of Booth et al. (2014). Our aim was to closely replicate the number line estimation findings of Booth et al. (2014); hence, we adopted their three number line estimation tasks. In addition, we included a fourth decimal number line task that was created by translating the fractions used in the fraction estimation task into magnitude equivalent decimals. Thus, a total of four scales were used: 0–1,000,000 whole numbers (12 trials), 0–62,571 whole numbers (12 trials), 0–1 fractions (18 trials), and 0–1 decimals (18 trials). The two measures of whole number magnitude (regular scale of 0–1,000,000 and atypical scale of 0–62,571) were combined to create a composite measure of whole number magnitude knowledge (as in Booth & Newton, 2012, and Booth et al., 2014).

The two whole number line tasks (replications of those used by Booth et al., 2014) were completed with a packet of 8.5 $\times$ 11-inch paper with a 20-cm line printed across the middle. The line was marked with 0 at the left end and 1,000,000 (or 62,571) at the right end. On each page, a number was written at the top and students were instructed to put a hatch mark on the line where that number would go. The numbers used were the same as those used by Booth et al. (2014). For the 0–1,000,000 scale, these were 3123, 7604, 12129, 20394, 85261, 132694, 298237, 358742, 453903, 595246, 724859, and 953271; for the 0–62,571 scale, the numbers were 19, 44, 176, 1059, 6426, 15023, 21649, 27393, 33691, 42672, 49126, and 54705.

The fraction number line task was identical to the whole number line tasks except that the scale of the number line was from 0 to 1 and numbers given were fractions rather than whole numbers. The fractions used were again identical to those used by Booth and colleagues (2014): 1/360, 1/180, 1/45, 5/118, 1/12, 13/85, 1/5, 3/11, 2/7, 1/3, 83/215, 177/352, 3/5, 5/8, 33/47, 7/9, 5/6, and 146/149.

The decimal number line task (a new task introduced in the current study) was identical to the fraction number Line task except that all of the fractions used in the fraction number line task were given as their decimal equivalents. The decimals varied in length from two to six digits, including some decimals with initial or terminating 0s, in order to avoid any perceptual cues that might lead students to interpret decimals using a whole number strategy. The decimals used were .00278, .0056, .022, .042373, .08, .1529, .200, .27, .28571, .3333, .386, .3324, .60, .62500, .702128, .78, .83333, and .980.

Students were randomly assigned to complete the four scales in one of six possible orders. The trials within each of the scales were randomized. Students were given the same instructions to complete the number line task as in Booth et al. (2014):

> "First we're going to work with four sets of number lines. You remember what a number line is, right? A number line is just a line with numbers across that shows us all of the numbers in order. In these number lines, only the numbers at the ends of the line will be marked, but not the ones in between. Your job is going to be to mark where you think some other numbers would go. Above each number line, there will be a number. Whenever you decide where you think the number goes, you need to place a mark on the number line where you think it goes. Make sure to pay attention to what the numbers are on the ends of the number lines because there are three different scales in your packet. Go ahead and work through your packet."

Similar to the procedure used by Booth et al. (2014), the distance between the left endpoint and the student's hatch mark on the number line was measured with millimeters (using a standard ruler). These measurements were then recorded on a computer and used to compute what the corresponding value would be at that hatch mark on the number line scale. These values were calculated as follows: (distance from left endpoint/total distance) $\times$ length of number line scale. For example, if a student placed a hatch mark for 2500 on the 0–1,000,000 line at 40 mm, the corresponding value would be (40/2000) $\times$ 1,000,000 = 200,000 (i.e., the student placed 2500 where 200,000 should go). These estimated values were plotted against the actual values that were given, and the R^2_{lin} for each participant was calculated. In addition, the percentage absolute error (PAE) was computed for each participant as follows: [(estimate − actual)/scale] * 100, following the procedure used by Booth et al. (2014).

Fraction relations task

This measure (a novel addition in the current study) consisted of a range of questions that measured students' understanding of different relational uses for fractions (see Appendix A). For example, questions measured knowledge of fraction equivalence, division, inverse, multiplying by the reciprocal, and identifying part-to-part ratios versus part-to-whole ratios in countable sets. These questions were taken from previous published work and research projects designed to investigate conceptual rather than procedural understanding of fractions. (Sources are noted in Appendix A.) Importantly, for all of the problems, there was no need to calculate the magnitude of any fraction. These questions, therefore, were quite distinct from the pure measure of magnitude understanding provided by the fraction number line task. The fraction relations task consisted entirely of multiple-choice questions that were scored on a binary basis (0 for incorrect, 1 for correct).

Fraction procedures task

An additional measure focusing on knowledge of fraction-related procedures consisted of problems taken from the "declarative fraction knowledge" items used by Booth et al. (2014). These questions were mainly designed to measure understanding of how to manipulate fractions as they appear in equations (see Appendix A). For example, questions included asking students to identify what the next procedural step would be to solve a problem. This measure was designed to control for connections between fractions and algebra that are based on simply knowing how to perform particular procedures that are common between fractions and algebra (e.g., manipulation of fractions, division operation). To solve all of these questions, participants must be able to identify correct procedures that are used when algebra problems involve fractions.

The questions assessing knowledge of fraction relations and fraction procedures were intermixed within a single battery, with questions presented to each student in one of six random orders. All of the questions used in the battery are shown in Appendix A.

Algebra knowledge task

Algebra knowledge was measured with a variety of questions (see Appendix B). Some of the questions were adapted from Booth and colleagues' (2014) measure of algebra knowledge, which included equation solving and "feature knowledge"—that is, understanding of properties of algebraic equations (e.g., is $4x - 3$ equivalent to $3 - 4x$?). This task also included three algebra word problems and other algebra problems taken from a bank of questions involving understanding and creating algebra expressions used in Algebra 1 courses.[1] Of the 14 total questions, 6 were in multiple-choice format and scored on a binary basis (0 for incorrect, 1 for correct). The remaining 8 items were equation-solving and word problem questions that were scored as correct if the student found the correct value for the missing variable and as wrong otherwise. No partial credit was awarded if the student wrote out the correct procedure but failed to find the correct answer.

Procedure

Students received all of these measures as pencil-and-paper tests in the following order: (a) number line estimation, (b) fractions (relational and procedural tasks), and (c) algebra knowledge (similar to the procedure followed by Booth et al., 2014, and Booth & Newton, 2012). The fixed order was used mainly due to practical limitations of controlling the packet completion in a classroom setting. Students were given approximately 50 min to complete all three packets, although most took less than 30 min. Students were not allowed to use calculators to complete the problems and were encouraged to write out their work on the paper.

Results

Distributions of performance for each measure

Multiple regression analyses were performed to identify those tasks that reliably accounted for unique variance in algebra performance. As a prerequisite to these analyses, we examined histograms and standard descriptive statistics of performance for each measure to ensure that all measures showed a reasonable degree of variability across our participants. For the number line tasks, the patterns of performance were very similar using either R^2_{lin} or PAE to assess level of performance. These two measures have often been used interchangeably in previous number line estimation studies (Booth et al., 2014; Siegler & Booth, 2004). For simplicity, we report only results based on PAE. The average PAE score for both the fraction number line (FNL) and decimal number line (DNL) measures was 15%. Variance in performance for DNL was slightly higher ($\sigma^2 = 122$) than that for FNL ($\sigma^2 = 101$). The average PAE score for the whole number line (WNL) measure was nearly identical at 16%, but variance in performance was considerably smaller ($\sigma^2 = 49$). Average proportion correct on fraction procedures questions was .57 ($\sigma^2 = .04$). Performance on fraction relations questions was slightly lower ($M = .42$, $\sigma^2 = .05$). Performance on algebra knowledge questions was also in the same range ($M = .48$, $\sigma^2 = .02$). All of the measures showed substantial variability in performance across students, satisfying a prerequisite for serving as potential predictors.

Intercorrelations among all measures

Table 1 shows the raw correlations among the three number line measures (WNL, FNL, and DNL), the fraction relations and fraction procedures measures, and the algebra knowledge measure. Note

[1] A sample of problems was taken from a pool of problems used by Belinda Thompson in work performed for her PhD dissertation (in progress) at the University of California, Los Angeles, which she kindly allowed us to use in the current study.

Table 1
Raw correlations between measures.

	Fraction relations	Fraction procedures	Fraction number line	Decimal number line	Whole number line
Fraction procedures	.38**				
Fraction number line	−.38**	−.19			
Decimal number line	−.45***	−.27*	.50***		
Whole number line	.07	−.07	.08	.03	
Algebra performance	.47***	.33**	−.39**	−.58***	−.14

* $.01 < p < .05$.
** $.001 < p < .01$.
*** $p < .001$.

that correlations of the number line measures with the other measures are negative because PAE is a measure of error. As observed by Booth et al. (2014), the PAE for FNL was significantly correlated with algebra performance ($r = -.39$, $p = .001$), whereas the PAE for WNL was not ($r = -.14$, $p = .26$). In addition, based on the new measures introduced in the current study, we found that the PAE for DNL ($r = -.58$, $p < .001$), performance on fraction relations ($r = .47$, $p < .001$), and performance on fractions procedures ($r = .33$, $p = .008$) were all significantly correlated with algebra performance. (Again, note that significant negative correlations indicate that higher performance on number line tasks is associated with higher performance on other tasks.)

Multiple regression analyses

Because several predictor variables showed reliable intercorrelations, we performed a series of multiple regression analyses to distinguish among the different predictors and to identify those that best accounted for unique components of variance in algebra performance. We first used the three number line measures as predictors in order to distinguish among these three measures of magnitude processing. The predictors were entered first as FNL, then DNL, and then WNL (because Booth et al., 2014, found FNL to be the leading predictor). The overall model accounted for a significant amount of variance in algebra performance, $F(3,61) = 11.50$, $p < .001$; however, only DNL contributed a significant proportion of variance: DNL, $\beta = -.51$, $t(61) = 4.28$, $p < .001$; WNL, $\beta = -.12$, $t(61) = 1.15$, $p = .27$; FNL, $\beta = -.14$, $t(61) = 1.14$, $p = .26$. We repeated the regression analysis using a different order of variables (DNL, then FNL, and then WNL) and obtained the same pattern of results.

A second set of regressions was then performed, including the fraction relations and fraction procedures measures in addition to the three number line estimation measures. The fraction relations predictor was inserted first because we hypothesized that this would be a strong predictor of algebra performance. Given that DNL was shown to be the most predictive magnitude measure, we then entered DNL, then FNL, and finally WNL. The last predictor entered was fraction procedures, which we did not expect to be closely related to algebra performance. The model, which is depicted in Fig. 1, accounted for a significant amount of variance in algebra performance, $F(5,59) = 8.64$, $p < .001$. However, only scores on the DNL task, $\beta = -.42$, $t(59) = 3.43$, $p = .001$, and the fraction relations task, $\beta = .24$, $t(59) = 2.12$, $p = .03$, accounted for a significant proportion of variance over and above the other three predictors: WNL, $\beta = -.13$, $t(59) = 1.31$, $p = .19$; FNL, $\beta = -.08$, $t(59) = 0.69$, $p = .50$; fractions procedures, $\beta = .11$, $t(59) = 1.03$, $p = .31$. We also tested the model by entering the magnitude measures first, followed by fraction relations and fraction procedures, and obtained the same pattern of results. Thus, among all of the predictor measures that were examined, only fractions relations and DNL performance predicted unique components of variance in algebra performance while controlling for the other measures.

Discussion

The current study provides evidence that both understanding of decimal magnitudes, as assessed by a number line task, and relational understanding of fractions are strong predictors of algebra

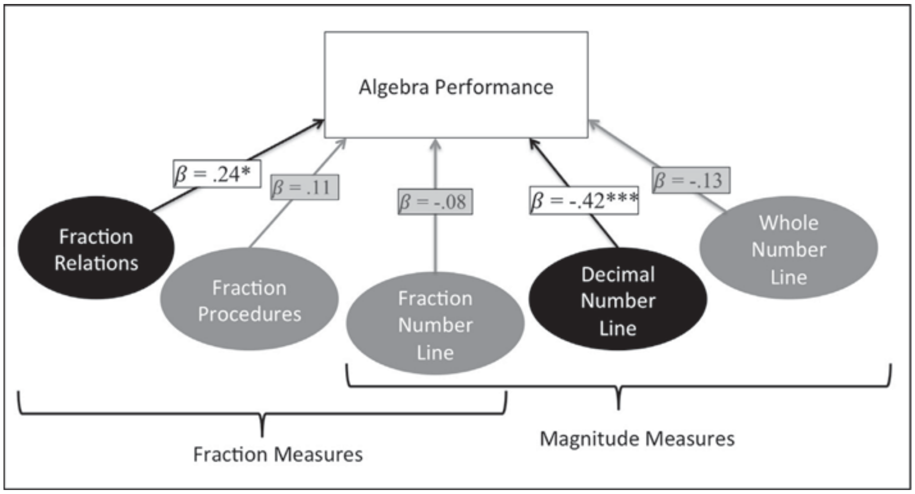

Fig. 1. Predictive model for algebra performance derived from multiple regression analyses.

performance. We replicated the empirical finding of Booth et al. (2014) and Booth and Newton (2012) that accuracy in number line estimation with fractions is related to algebra performance (whereas accuracy with whole numbers is not); however, multiple regression analyses revealed that this linkage no longer holds when performance with decimals on the same task is considered. In addition, we showed for the first time that a different and distinct aspect of fraction knowledge—a measure of understanding relations involving fractions—adds a unique contribution to predicting algebra performance. Together, measures of decimal magnitude understanding and relational fraction understanding reliably predict early algebra performance.

Many previous studies have focused primarily on number line estimation and its predictive power. Successful performance on the fraction number line task appears to involve two components: (a) performing a division operation based on the relation between the numerator and denominator and (b) approximating the resulting magnitude on the physical line. Our results suggest that these two skills can be distinguished by separate measures. The former can be assessed by a test of relational knowledge with fractions, and the latter can be assessed by the number line task using decimals (because decimals obviate the need to perform division). These two specific measures provide more accurate predictors of algebra performance than the fraction number line task.

Further research is needed to understand why the number line task with decimals predicts algebra performance more effectively than the same task with other number types. As rational numbers, both fractions and decimals are more complex than whole numbers. But as noted above, number line placement with decimals may provide a "purer" measure of magnitude comprehension than the same task with fractions because decimal magnitudes can be accessed without performing a division operation. Interpreting a multi-digit decimal value certainly requires a form of relational processing, as does interpreting a multi-digit whole number. Decimal notation is distinct from that of whole numbers (particularly because a leading zero can appear to the right of the decimal point and to the left of any other integers). However, research on magnitude comparisons has shown that performance with decimals is similar in accuracy and speed to performance with multi-digit whole numbers, whereas comparisons with fractions are much more difficult (DeWolf et al., 2014; Huber, Klein, Willmes, Nuerk, & Moeller, 2014; Iuculano & Butterworth, 2011). Thus, it seems that decimals, like whole numbers but unlike fractions, can be readily interpreted as expressing a one-dimensional magnitude.

However, there are other possible interpretations for the greater predictive power of the decimal version of the number line task. Decimal performance was somewhat more variable than performance with other number types. This greater variability might reflect the fact that decimals are introduced to students later than fractions and, hence, are the number type with which the students in our study had the least experience. Consequently, the decimal number line estimation task may simply be measuring general math ability (because the more precocious students may have mastered decimal magnitudes earlier than their age-matched peers). Previous research has shown that as number line estimation tasks (with whole numbers or fractions) become progressively more difficult (by increasing the size of the scale, using unusual scales, or including fractions), performance on these tasks is correlated with the overall development of general math ability (Booth & Siegler, 2008; Ramani & Siegler, 2008; Siegler et al., 2011). Further research is needed to determine whether decimal number line estimation is especially correlated with overall math knowledge.

An equally intriguing finding from the current study is that understanding of fraction relations is a reliable additional predictor of algebra performance, separable from measures of magnitude understanding. Relational understanding of fractions, thus, appears to provide a stepping-stone toward acquiring the cognitive skills needed to form and understand algebraic expressions. The bipartite format of fractions sets them apart from all other number types, enabling them to convey relations between sets more effectively than their decimal magnitude counterparts (DeWolf et al., 2015). Thus, fractions provide an early opportunity for students to understand the concept of expressing relations between quantities. In fact, excessive emphasis on understanding fraction magnitudes may obscure their relational meaning.

In particular, it appears that this type of relational understanding might be most useful for understanding algebraic expressions. Understanding how to create and manipulate algebra expressions is a crucial aspect of mastering algebra and is important for solving word problems and equations. In general, understanding how to appropriately construct and manipulate fraction expressions is necessary for successful construction and manipulation of algebraic expressions. However, this linkage goes beyond simply being able to perform the same rote procedures given that success on the common procedures measured by the fraction procedures questions did not uniquely predict algebra performance. Based on this finding, it seems that algebra instruction should be related more closely to fraction instruction in order to bootstrap students' understanding of algebra.

Other researchers have pointed out the connections between understanding relational concepts in arithmetic and understanding those in algebra. Empson and colleagues (Empson & Levi, 2011; Empson, Levi, & Carpenter, 2011) suggested that students' basic intuitions about arithmetic functions (across both whole numbers and fractions) can be exploited to build basic relational concepts linking arithmetic to algebra. In particular, learning about fraction relations may help students to acquire some implicit understanding of general regularities such as the associative property. For example, to add $3/4 + 1/2$, a student might reason that $3/4$ is equal to $1/2 + 1/4$. Accordingly, one can add $1/2 + 1/2$ to get 1 and have $1/4$ left over (Empson, 1999). Empson (1999) argued that fraction learning can be connected to basic understanding of properties of algebra (e.g., the use of the distributive property of multiplication over addition to add $7a + 4a$ to get $11a$, which is similar to the reasoning required to understand how to add $70 + 40$ or $7/5 + 4/5$).

Fractions are the first example of numbers that are relational expressions to which students are exposed. Their format has implications for the types of procedures that are appropriate for students to perform. Taking the example from Empson and colleagues used above (Empson & Levi, 2011; Empson et al., 2011), one could consider $7a$ to be an expression of 7 units each of size a. This interpretation is similar to the interpretation of the fraction $7/5$ as an expression of 7 units of $1/5$. Such an understanding of $7/5$ has implications for the appropriate ways to perform arithmetic operations such as grasping the constraint that "unlike terms" cannot be combined in algebra (e.g., $7a + 8b$ does not equal $15ab$, just as $7/5$ and $8/9$ cannot be combined to get $15/45$ or $15/14$). As students become fluent with the concept of a relational expression, their understanding of fraction relations may help to bootstrap their learning about similar relations within algebra. Thus, as the current findings suggest, a conceptually rich understanding of fractions may be especially important for understanding algebra.

The current research establishes a correlation between fraction understanding and algebra performance and shows that this factor is separable from the predictive power of measures of magnitude. However, further research is required to determine whether these linkages between rational number knowledge and learning of algebra are causal. Future studies should test whether direct instruction in relations involving fractions improves algebra performance over and above the potential usefulness of instruction focusing on magnitudes of rational numbers. A program of research aimed at assessing the effectiveness of alternative instructional interventions would help to determine whether fraction understanding has a causal impact on algebra performance.

Currently, recommendations to educators highlight the importance of teaching students the magnitudes of fractions and especially emphasize use of the number line representation to highlight relative magnitudes (National Mathematics Advisory Panel, 2008; Siegler et al., 2010). The current findings suggest that training placement of decimals on a number line representation may be at least as effective for instruction. Of equal importance, teachers may need to focus on highlighting the connections between fraction and algebraic expressions, thereby capitalizing on the relational parallels between these two important domains of mathematical knowledge.

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Appendix A. Appendix

A.1. Fraction problems

Procedural fraction questions (from Booth et al., 2014, "declarative fraction questions")
Is either of these an effective first step toward solving for z in the equation $3 = \frac{1}{z}$?
Circle yes or no.

(a)	Multiply both sides by z.	Yes	No
(b)	Divide both sides by 3.	Yes	No

Would any of the following steps be an effective first step toward solving the equation $\frac{6}{d} = 2$?
Circle yes or no.

(a)	Subtract 2 from both sides.	Yes	No
(b)	Multiply both sides by d.	Yes	No

If $y = 3x + 2$, which of these expresses x in terms of y? Circle the correct answer.

a. $x = \frac{y-2}{3}$ b. $x = \frac{y+2}{3}$ c. $x = \frac{y}{3} - 2$ d. $x = \frac{y}{3} + 2$

On Planet Zebula, zeds are a unit of money. On this planet, Carla paid x zeds for three cartons of juice. What is the price in zeds of one carton of juice? Circle all that apply.

a. $\frac{x}{3}$ b. $\frac{3}{x}$ c. $3 + x$ d. $\frac{1}{3}x$ e. $3x$

A.2. Relational fraction questions

A.2.1. Multiplicative/Division relations

(adapted from B. Thompson, 2014, unpublished dissertation)

Which expression shows a way to find half a number, n? Circle yes or no for each expression:

$n \div \frac{1}{2}$ yes no

$n - \frac{1}{2}$ yes no

$n \div 2$ yes no

$n \times \frac{1}{2}$ yes no

(adapted from Brown & Quinn, 2006)
What is $\frac{7}{\frac{3}{5}}$ equal to?

(a) $7 \times \frac{3}{5}$
(b) $\frac{3}{5} \div 7$
(c) $\frac{35}{3}$
(d) $\frac{21}{5}$

Inverse relation (adapted from Brown and Quinn, 2006)
(n is an integer greater than 0)
If n increases in value, then $1/n$

(a) gets very close to 1
(b) gets very close to 0
(c) increases in value, too

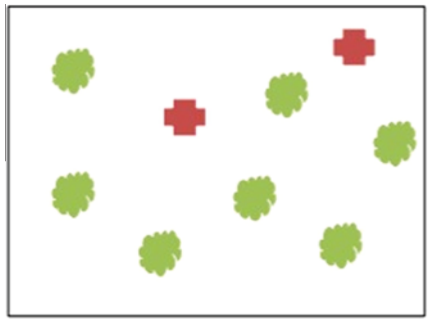
Equivalence relation (adapted from B. Thompson, 2014, unpublished dissertation)
Which fraction is equal to 15/20?

(a) 20/25
(b) 9/12
(c) 20/15
(d) none of these

Which fraction is equal to 8/12?

(a) 24/48
(b) 16/36
(c) 12/18
(d) none of these

Identifying ratio relations (adapted from DeWolf et al., 2015)
In the pictures below, there are two types of relationships: part-to-part ratio (number of crosses to number of clouds) and part-to-whole ratio (number of crosses to total number of crosses and clouds OR number of clouds to total number of crosses and clouds). Circle the relationship that each fraction represents:



$\frac{2}{7}$ (a) part -to-part ratio (b) part -to -whole ratio

$\frac{4}{18}$ (a) part -to-part ratio (b) part -to -whole ratio

$\frac{14}{18}$ (a) part -to-part ratio (b) part -to -whole ratio

$\frac{7}{9}$ (a) part -to-part ratio (b) part -to -whole ratio

Appendix B. Algebra problems

Equation solving (from Booth et al., 2014)
Solve the equations:

(1) $5 = x - 7$
(2) $\frac{8}{k} = 4$
(3) $-4x + 5 = 8$
(4) $\frac{6}{b} = 9$
(5) $-3y + 6 = 8 + 5y$

Feature knowledge (from Booth et al., 2014)
Which of the following is equal to $-4x + 3$?

(a)	4x + 3	yes	no
(b)	3 – 4x	yes	no
(c)	4x – 3	yes	no
(d)	3 + (–4x)	yes	no
(e)	3 + 4x	yes	no

If $10x - 12 = 17$ is true, which of the following must also be true?

(a)	10x – 12 + 12 = 17 + 12	yes	no
(b)	x – 2 = 17	yes	no
(c)	10x = 29	yes	no
(d)	10x = 17	yes	no
(e)	10x – 10 – 12 – 10 = 17	yes	no
(f)	10x – 12 + 12 = 17	yes	no

Word problem solving (from Booth et al., 2014)

The Carlson family is moving today. It took 89 boxes to pack up all of their things. Dad told each of his four children to carry 16 boxes to the truck and then he would get the rest. How many boxes did Dad have to carry?

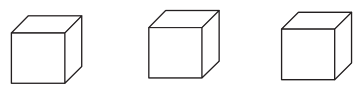
Tommy bought a pair of shoes on sale. It was 1/4 off the original price. He paid $42. What was the original price of the shoes?

(not taken from Booth et al., 2014)

Ted has $12 more than Carla, and Carla has $8 more than Devon. Together, Ted and Carla have four times as much as Devon. How much money does each person have?

Understanding algebra expressions (adapted from B. Thompson, 2014, unpublished dissertation)

Each of these three boxes weighs the same amount. If the weight of one box is x, what is the weight of the three boxes together?



$3x$
$x + 3$
3

It's impossible to tell

What could be the value of x that makes this equation true?

$x + x + x = 15$
5,5,5
4,5,6
3,3,9
all of these

If $a + b = c$ and $d + e = c$, when is $a + b = d + e$ true?

always
never
it depends on the values of a, b, c, d, and e

If n is some number, and k is 4 less than n, which expression represents k?

4
$n - 4$
$4 - n$
it's impossible to tell

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