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Relational Priming Based on a Multiplicative Schema for Whole Numbers and Fractions

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Abstract

Why might it be (at least sometimes) beneficial for adults to process fractions componentially? Recent research has shown that college-educated adults can capitalize on the bipartite structure of the fraction notation, performing more successfully with fractions than with decimals in relational tasks, notably analogical reasoning. This study examined patterns of relational priming for problems with fractions in a task that required arithmetic computations. College students were asked to judge whether or not multiplication equations involving fractions were correct. Some equations served as structurally inverse primes for the equation that immediately followed it (e.g., $4 \times 3/4 = 3$ followed by $3 \times 8/6 = 4$). Students with relatively high math ability showed relational priming (speeded solution times to the second of two successive relationally related fraction equations) both with and without high perceptual similarity (Experiment 2). Students with relatively low math ability also showed priming, but only when the structurally inverse equation pairs were supported by high perceptual similarity between numbers (e.g., $4 \times 3/4 = 3$ followed by $3 \times 4/3 = 4$). Several additional experiments established boundary conditions on relational priming with fractions. These findings are interpreted in terms of componential processing of fractions in a relational multiplication context that takes advantage of their inherent connections to a multiplicative schema for whole numbers.

Keywords: Mathematical cognition; Relational priming; Rational numbers

1. Introduction

1.1. Representation and processing of fractions

Recent years have seen a surge in research on the mental representation of fractions (typically one of the first new number formats that students encounter in school after the

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natural numbers, the concept of 0, and negative numbers). Although fractions are a part of the number system, they differ importantly from integers and decimals in their bipartite (*a*/*b*) structure. Researchers have largely focused on the issue of whether the magnitude of a fraction is retrieved as a holistic value or calculated by an on-line evaluation of the magnitudes of the numerator and denominator, considered separately. For example, Bonato, Fabbri, Umiltà, and Zorzi (2007) found evidence that, in performing a magnitude comparison task, adults do not process fractions holistically but rather assess the relative sizes of the numerators and denominators. In contrast, Schneider and Siegler (2010) found that adults *do* process fraction magnitudes holistically when the task does not allow them to use any shortcut strategies that would enable separate processing of the numerator and denominator magnitudes. Studies using other assessment methods have found variable evidence for holistic and componential processing (Ischebeck, Schocke, & Delazer, 2009; Jacob, Vallentin, & Nieder, 2012; Kallai & Tzelgov, 2012; Meert, Grégoire, & Noël, 2009), and it appears that people can use a variety of procedures to assess fraction magnitudes (Fazio, DeWolf, & Siegler, 2016). DeWolf, Grounds, Bassok, and Holyoak (2014) used a magnitude-comparison task to examine adults' magnitude assessment for fractions, decimals, and whole numbers. Like Schneider and Siegler (2010), they found evidence of holistic processing of fraction magnitudes. Importantly, they also found that the processing of fractions magnitudes was much more effortful and error-prone relative to processing the magnitudes of either decimals or whole numbers.

Researchers have suggested that componential processing of fractions is evidence of a "whole number bias," indicating that adults and children have not fully mastered the concept of fractions (Ni & Zhou, 2005). Thus, when students appear to have difficulty representing holistic magnitudes of fractions, this is taken as evidence of poor understanding or misconceptions (Stafylidou & Vosniadou, 2004). However, an important question is whether it might (at least sometimes) be *beneficial* for adults to process fractions componentially. Some previous studies have already identified situations in which adults process fractions componentially, especially when easy shortcut strategies that do not require holistic processing are available (Fazio et al., 2016; Ischebeck, Weilharter, & Körner, 2016; Obersteiner et al., 2013).

Although fractions are clearly less effective than decimals in conveying number magnitudes, recent research has shown that the bipartite format of fractions can be advantageous for reasoning about *relations* between quantities. DeWolf, Bassok, and Holyoak (2015a) had adults identify one of two possible set relations in a visual display (part-to-whole or part-to-part), which were denoted by either fractions or by magnitude-equivalent decimals (e.g., 3/7 or 0.43). When the display showed countable entities, participants performed the task more quickly and more accurately when the number representing the target relation was a fraction rather than a decimal. These findings indicate that the componential (bipartite) structure of fractions is better suited for a relational reasoning task than is the one-dimensional format of decimals.

The existence of advantages for componential processing for fractions would be consistent with evidence that other types of multi-digit numbers (e.g., multi-digit integers) are typically processed in a componential fashion (e.g., Huber, Moeller, Nuerk, &

Willmes, 2013; Huber, Nuerk, Willmes, & Moeller, 2016). In this study, we explored the potential benefits of componential fraction processing, focusing on a task that requires arithmetic computations (rather than comparison of numbers to perceptual displays, as in DeWolf et al., 2015a).

1.2. Multiplicative reasoning with whole numbers and fractions

One task in which componential representation of fractions would appear to be extremely useful is in performing multiplication and division with fractions. For these two arithmetic operations, fraction numerators and denominators can be manipulated separately along with other multiplicatively related whole-number and fraction components in a problem (e.g., 3/4 × 1/3 can be simplified to 1/4 by thinking of the problem as 3/3 × 1/4; i.e., in essence "crossing out" 3 in the numerator of the first fraction and 3 in the denominator of the second). Within the context of a fraction multiplication problem, thinking of a fraction as a holistic unit might actually lead students astray. For example, applying the common adage "multiplication makes bigger" to the problem 31/56 × 17/42 might lead to the conclusion that the resulting product will be larger than either of the two multiplicands. And, in fact, middle-school students and preservice teachers reliably conclude that the product in such problems will be larger (Siegler & Lortie-Forgues, 2015). However, these same students are able to correctly solve multiplication problems with fractions. This pattern suggests that people's ability to solve a fraction problem, or correctly apply procedures, is dissociable from their understanding of the magnitude of the result.

To be successful in fraction arithmetic, students must understand how numbers are multiplicatively related (Behr, Lesh, Post, & Silver, 1983; Kieren, 1976; for a review see Ohlsson, 1988). For example, a skilled understanding of fractions would involve knowing that 6/8 can be reduced to 3/4 because both 6 and 8 can be evenly divided by the same number (their common factor of 2). Because the numerator and denominator are both divided by the same number, the internal relationship between numerator and denominator is maintained. In contrast, 6/7 cannot be reduced because 6 and 7 do not share a common factor. Students are often encouraged to simplify fractions, a procedure that would be expected to highlight common factors linking numerator to denominator.

Understanding fractions as flexible (e.g., they can be simplified, can represent the division operation, or can be utilized componentially depending on the context) lies at the center of a truly generalizable understanding of multiplication itself. Thompson and Saldanha (2003) have argued that a more general understanding of multiplication goes beyond simply thinking of multiplication as repeated addition, which does not fit well with the differences in procedures across arithmetic operations with fractions. Rather, the reciprocal relation is a key. Understanding multiplication conceptually requires grasping multiple reciprocal relationships between the two factors, n and m, and their product nm (e.g., nm is n times as large as m; or conversely, n is $1/m$ times as large as nm). Thus, division with fractions is most generally understood as multiplying by the reciprocal, thereby uniting fraction multiplication and division as inverse operations on complementary problems. This construal is general because it is compatible with all semantic interpretations of these operators (e.g., partitive and quotative division in a variety of real-world situations).

Importantly, this general understanding of fraction multiplication and division (e.g., a fraction incorporates a division relation between numerator and denominator) is orthogonal to a meaningful semantic understanding of fractions as numbers. Fractions are relational in the sense that their notation makes certain relations more explicit than do other formats (e.g., decimals). For example, the fraction notations 2/3 and 4/6 support the application of various procedural rules to the numerator and denominator of each. This type of knowledge is distinct from semantic understanding of the relative magnitudes of the numerator and denominator (i.e., recognizing the numerical equality of the two forms of the fraction).

Research on whole number arithmetic has already shown that there are deep connections in the representation of multiplication and division "facts," such that the standard "times table" taught in school serves psychologically as a kind of *multiplicative schema* (Campbell, 1997, 1999a). Campbell's studies provided evidence that when adults are solving division problems, they do so by inverse multiplication. For example, when solving $12 \div 4$, the solver might reformulate the problem as $4 \times ? = 12$, and activate all the factor pairs with a product of 12 (1×12, 2×6, 3×4). The solver could then select 3, the complementary factor to 4, as the answer. Campbell found that when given a multiplication problem first (e.g., 7×9), adults were faster to solve a corresponding division problem ($63 \div 9$) than when they were given the division problem without the multiplication problem as a prime (see also Mauro, LeFevre, & Morris, 2003; Verguts & Fias, 2005). In this paper, we aim to investigate to what extent adults integrate fractions into their whole-number multiplicative schema. That is, do adults treat a fraction flexibly as a division relationship between component parts, which can be manipulated depending on the context, or do they view the fraction only as a holistic number?

1.3. Math expertise and relational processing

Recent research suggests that grasping multiplicative relations between numbers depends on degree of math expertise. For example, Murphy, Rogers, Hubbard, and Brower (2015) found that adults rank 2 as more similar to 4, 6, and 8 than to 3, presumably because 2, 4, 6, and 8 are all multiples of 2. This type of relational understanding seemed to be dependent on expertise, however, as Murphy et al. (2015) found that non-experts based numerical similarity primarily on magnitude. Such differences in similarity judgments between non-expert and expert adults suggest that a more developed understanding of number creates relational similarity between numbers based on additional factors, such as multiplicative relations, rather than solely on magnitudes of individual numbers.

The contrast between non-expert versus expert judgments of number similarity echoes other expert-novice differences that have been identified in the area of mathematical cognition. Expert mathematicians (e.g., math professors, or those who achieve high scores on a math proficiency test) are more likely to judge problems embodying the same mathematical structure to be similar, are more likely to apply the same problem-solving strategies to relationally related problems, and demonstrate greater transfer after a delay (Novick, 1988; Novick & Holyoak, 1991).

Understanding of fractions provides a particularly interesting context for investigating differences in expert and novice understanding. Despite fraction instruction in elementary school (and beyond), magnitude misconceptions seem to persist in older students and adults with lower overall competence in mathematics. As noted earlier, in the realm of (positive) whole numbers, multiplication yields a larger result and division a smaller result; but this is not generally true of rational numbers, such as fractions or decimals <1. While Siegler and Lortie-Forgues (2015) found that participants from middle school and preservice teachers made systematic errors when deciding whether problems like $31/56 \times 17/42 > 31/56$ are true, math and science students from a highly competitive university were consistently correct. One potential explanation for such errors is that negative transfer occurs for relative novices: Middle-school students and preservice teachers are incorrectly transferring their schema of multiplication from positive whole numbers to rational numbers <1. The persistence of expert-novice differences in fraction arithmetic suggests that instruction in elementary school (and beyond) is not sufficient for a variety of students.

The lack of understanding by novices is especially evident in the case of the "invert and multiply" strategy in fraction division. Early on, students are taught that to complete a fraction division problem, all that is required is to invert the second fraction in the problem and then proceed with the fraction multiplication procedure. But understanding *why* this strategy works is not simple. Tirosh (2000) found that even preservice teachers have little understanding of this strategy. The reason why the invert-and-multiply strategy works involves the reciprocal relationships between the two factors in a multiplication problem and their relationship to a product. For example, $5 \div 10$ is the same as 5/10, which is the same as $5 \times 1/10$ since 10 and 1/10 are reciprocals. In addition, students have to understand that the "bar" in the fraction expression denotes a division operation; and as such, it can be used to represent a relation within the multiplication operation itself. Thus, in fact, the same strategy applies when multiplying either whole numbers or fractions. Performing the operation "$2 \div 3$" is equivalent to "2/3" and also "$2 \times 1/3$." Furthermore, "$2 \div 3$" represents the same proportional relation as "$4 \div 6$." A general understanding of this relational structure would allow students to move flexibly between any of these equivalent notations.

1.4. This study

If adults are aided in solving division problems by inverse multiplication (Campbell, 1997, 1999a; Campbell & Alberts, 2010), then perhaps an expert understanding of fractions (a form of division) is also grounded in the multiplicative schema (a kind of mental "times tables"). This study aims to explore this possibility, extending the conception of a multiplicative schema as introduced by Campbell (1997, 1999a). We propose and test a model of how a multiplicative schema might operate to produce a kind of relational priming for multiplication problems involving fractions. In addition, we investigated how relational priming in multiplication with rational numbers varies with overall math understanding.

To this end, we explored a relational priming paradigm based on pairs of *equation inverses*, each consisting of a prime and target equation that are inversely related to one another (e.g., $3 \times 4/3 = 4$ preceded by $4 \times 3/4 = 3$).[1] Experiment 1 tests whether adults from a highly competitive university show facilitation in verifying a multiplication equation when it is preceded by an inversely related equation. We hypothesized that participants would show priming for such fraction equations.

The subsequent experiments used the same paradigm but varied the stimuli. In Experiment 1 there was a great degree of perceptual overlap between the two equation inverses (each of the component numbers completely matched). Experiment 2 introduced equations for which the numbers did not match *between* equation inverses (e.g., $3 \times 8/6 = 4$; $4 \times 3/4 = 3$). Experiment 2 also assessed how the priming effect may differ as a function of math expertise, by comparing students who were relatively low or high in general math ability. We hypothesized that both low- and high-performing students will show priming for equations with high perceptual similarity, but that only high-performing students will show priming for the equations in which such similarity is reduced, so that numbers match only relationally.

Experiments 3A, 3B, 4, and 5 were all designed to further assess whether the priming effect (found with high-performing participants in Experiment 2) is indeed relational in nature, and not simply the product of perceptual fluency aided by a match between equations. In Experiment 3A, we varied the order of the whole number and fraction elements across equations in a pair (e.g., $3 \times 4/3 = 4$; $3/4 \times 4 = 3$). We hypothesized that a superficial change that preserves the reciprocal fraction relationship between equations (i.e., one that solely changes where the fraction relationships are located in the equation) would not diminish the priming effect. In Experiment 3B, we changed the order of the individual elements (numerator and whole number) between equations to create a relational mismatch between equations (e.g., $3 \times 4/3 = 4$; $3 \times 4/4 = 3$). For these equation pairs, we hypothesized that the relational mismatch would prevent priming because this change does not preserve the reciprocal fraction relationship between equations (e.g., 4/4 is not a reciprocal of 4/3).

In Experiment 4, we assessed the extent to which participants treat these fraction equations as division equations in which the individual elements of the fraction can be manipulated separately. We hypothesized that when equations are arranged in a division format, participants will show a priming effect similar to that obtained with fractions, because both of these formats highlight the shared relational structure between prime and target. Finally, Experiment 5 tested whether a priming effect will be found for fractions that are reciprocals in magnitude but are not divisible by one another (e.g., the reciprocal pair 24/9; 6/16 can be reduced to 8/3 and 3/8, respectively; however, 24 cannot be simplified to 16 and 9 cannot be simplified to 6). On the basis of a process model for the task (described below), we hypothesized that in these cases no priming will be observed.

2. Experiment 1

The goal of Experiment 1 was to first establish whether the equation inverse relation across pairs of fraction multiplication equations (e.g., 3 × 4/3 = 4; 4 × 3/4 = 3) can serve as a useful basis for relational priming for skilled adults when the prime and target equations are maximally similar.

2.1. Method

2.1.1. Participants

Participants were 30 undergraduates from the University of California, Los Angeles (UCLA; M_{age} = 20; 22 females) who received course credit. As we will show in Experiment 2, students drawn from this population generally have relatively good skill in basic arithmetic.

2.1.2. Design and materials

Equation type (prime vs. target) design was varied as a within-subjects factor. All stimuli consisted of multiplication equations in the form: $a \times [fraction] = b$. Participants were told to estimate whether the equation answer was true (correct) or false (incorrect), rounding to the nearest whole number.[2]

For these fraction equations (e.g., 3 × 4/3 = 4), we intended for participants to consider the fraction as a separate perceptual unit (e.g., Landy & Goldstone, 2010) that would be processed distinctly (as compared to a format using whole numbers only, such as (3 × 4)/3 = 4). Specifically, we formatted the equations to have the whole number separated with a space between the multiplication operator and the fraction (which contained no spacing around the division operator) so that participants would view the whole number and the fraction as two separate chunks being multiplied. However, there is a possibility that participants could have processed the equations according to the formal order of operations (i.e., PEMDAS: Parentheses, Exponents, Multiply, Divide, Add, Subtract), rather than treating the fraction as a separate unit. To evaluate this possibility, we showed the equations to pilot participants and asked several questions about how they would solve the equation and whether there appeared to be a fraction in the equation. Pilot participants agreed that the equation should be viewed as a whole number times a fraction, but that the component parts of the fraction could be manipulated independently. During the actual experiment, participants were told that the equations in the experiment were "multiplication" equations to encourage them to view the equation as involving two distinct factors being multiplied.

Equations were presented using Arial sans serif font as shown in Appendix Table S1 in the Supporting Information. Equations were shown sequentially, and unbeknownst to participants, were arranged in two kinds of pairs: *primed* pairs and *foil* pairs. There were a total of 120 pairs (or 240 total equations). The pairs were evenly divided into 60 prime pairs and 60 foil pairs. The primed pairs consisted of prime and target equations that were related to one another by virtue of a shared equation inverse relation. Of these primed pairs, 30 pairs were true (i.e., correct) and 30 pairs were false (incorrect). The foil pairs consisted of a variety of equation types that did not share the equation inverse relation. These foils were constructed to block potential short-cut strategies. (See Appendix Table S1 for a summary of such potential strategies.)

The true primed pairs in Experiment 1 were in the basic format: $a \times b/a = b$, followed by $b \times a/b = a$. This is the simplest case of the general formula for equation inverses: $a \times b/c = d$; $d \times c/b = a$. The pairs used in Experiment 1 are termed *Complete-Match* pairs because within both the prime and target equation, the numbers used in the numerator and denominator of the fraction, respectively, match the whole-number product and multiplicand; in addition, the numbers in the numerator and denominator of the prime are identical to those in the target equation, with roles reversed. Table 1 provides examples of the types of true primed pairs used in all experiments, and Appendix Table S1 lists all stimuli. In Table 1, the Complete-Match example is $3 \times 4/3 = 4$ followed by $4 \times 3/4 = 3$. In each of the subsequent experiments, we varied the degree of difficulty of the true primed pairs by adding an additional between-subjects condition. These additional types of true primed pairs will be described within the Method for each experiment.

Within each pair, the order of presentation of each equation was randomized, and across all pairs, the order of presentation was randomized for each participant. Appendix Table S1 lists all of the stimuli used for equations in Experiments 1–5, and Appendix Table S2 briefly describes the nature of the various types of foils.

2.1.3. Procedure

In Experiment 1 and all subsequent experiments, stimuli were displayed with Macintosh computers using Superlab 4.5 (Cedrus Corporation, 2004), and response times (RTs) and accuracy were recorded. Participants were instructed that they would need to decide whether a series of multiplication equations were correct or incorrect. If the equation was correct, they were instructed to hit the *a* key; if it was incorrect, they were to hit the *l*

Table 1
Examples of the true primed pair types used in each experiment

Pair Type	Example
Complete Match (Experiments 1, 2, and 5)	$3 \times 4/3 = 4$ $4 \times 3/4 = 3$
Between-Equation Mismatch (Experiment 2)	$3 \times 4/3 = 4$ $4 \times 6/8 = 3$
Within-Equation Mismatch (Experiments 3A, 3B, and 4)	$3 \times 8/6 = 4$ $4 \times 6/8 = 3$
Double-Mismatch-D (Experiment 5)	$8 \times 6/4 = 12$ $12 \times 2/3 = 8$
Double-Mismatch-ND (Experiment 5)	$2 \times 21/6 = 7$ $7 \times 4/14 = 2$

key. As we were particularly interested in potentially subtle RT differences, participants were instructed to respond as quickly as possible while maintaining high accuracy. There was no time limit for responding. Participants were first given four practice equations that used only whole numbers. After the practice equations, they were given a chance to ask questions before starting the test equations.

2.2. *Results and discussion*

2.2.1 *Accuracy*

Accuracy for each participant was averaged for each equation of true primed pairs. Mean accuracy values for primed pairs (i.e., the prime equation and the target equation) are shown in Fig. 1 (left). There was no priming effect for accuracy (95% accuracy for both prime and target).

2.2.2. *Response times*

Response times for each participant were averaged over each equation for the true and false primed pairs. Response times for incorrect answers were excluded from these analyses. Mean RTs are shown in Fig. 1 (right). For Complete-Match equations, the target was solved significantly more quickly than the prime (prime: 2.58 s vs. target: 2.21 s; $t(29) = 3.08$, $p = .004$). Thus, RTs showed a significant speed-up consistent with a priming effect.

To assess whether the pattern of priming only held for the true primed equations, we also performed a comparison of the corresponding false prime and target equations. There was no significant priming effect (prime: 3.89 s vs. target: 3.79 s; $t(29) = .61$, $p = .55$). Thus, the priming effect on RTs for Complete-Match fractions was only observed when the equations were true.

In addition, there was no priming effect for the True/True Cross-out foils (prime: 2.35 s vs. target: 2.55 s; $t(29) = 1.58$, $p = .13$). Overall, the lack of priming effects for

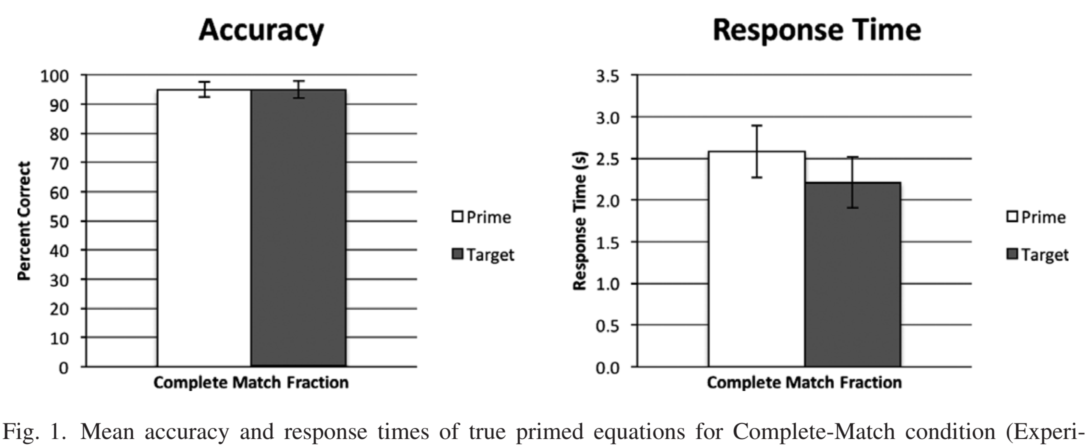

Fig. 1. Mean accuracy and response times of true primed equations for Complete-Match condition (Experiment 1).

false primed pairs and for the various types of foils rules out several short-cut strategies (see Appendix S1) that might otherwise have offered alternative explanations for the priming effect observed with Complete-Match true fraction pairs.

2.2.3. *Analysis of response time distributions*

To better understand the nature of the priming effect observed for the Complete-Match fractions, we conducted a modified Vincentile response-time distribution analysis (see Balota, Yap, Cortese, & Watson, 2008; Hoedemaker & Gordon, 2014; Yap, Balota, & Tan, 2013). This analysis focuses on the distribution of RTs for prime and target equations to determine if the effect is characterized by a general shift across equations across the entire distribution, or whether it selectively arises for equations drawn from some selective portion of the distribution (e.g., only those equations solved relatively slowly, presumably because of their greater difficulty; Balota et al., 2008). A prediction of our process model (see below) is that priming will involve a general speed-up of responses to target equations. Hence the process model predicts that the priming effect will appear across the entire range of the response-time distribution. In contrast, if a speed-up is only found for relatively slow responses (i.e., difficult equations), this pattern would suggest that priming is retroactive, being strategically invoked when a target equation proves to be difficult (Yap et al., 2013).

The Vincentile analysis is performed at the level of individual participants. Because we had a relatively small number of primed pair equations (30), we divided the response-time distribution for these items into six bins (rather than the traditional 10) to provide adequate stability. Response times for primes and targets for each participant were trimmed to exclude any outside of three standard deviations from the mean. Response times were broken down into six bins (the fastest 16.67%, the next fastest 16.67%, and so on). The means of the six bins for each participant were then averaged across all participants (see Fig. 2)

The predicted general shift in RTs across the whole distribution implies a main effect of equation type (prime vs. target) and no interaction with bin number. As hypothesized, the Vincentile analysis revealed that target equations were overall faster than prime equations (prime: 2.05 s vs. target: 1.77 s; $F(1, 27) = 14.06$; $MSE = .49$; $p = .001$), and no reliable interaction between bin number and equation type ($F(5, 135) = 1.39$; $MSE = .10$; $p = .23$). The Vincentile analysis thus indicates that priming was obtained consistently across the full distribution of RTs, rather than being specific to some selective range, such as the slowest responses (i.e., most difficult equations).

3. A process model of priming within a multiplicative schema

The results of Experiment 1 indicate that adults are faster and more accurate overall in evaluating multiplicative equations when these are based on Complete-Match fractions. These findings suggest that college students exploit their knowledge of multiplicative facts and therefore are sensitive to inverse relations between equations expressed with

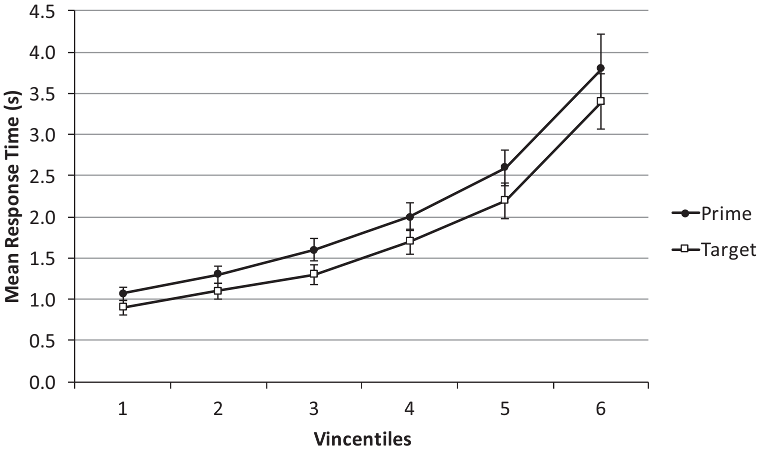

Fig. 2. Distribution of response times across six Vincentiles for true prime and target Complete-Match fraction equations (Experiment 1).

fractions (at least when the constituent numbers in the successive equations are identical). The experiments reported below were designed to assess the limits of relational priming based on a multiplicative schema, as a function of math expertise (Experiment 2) and the form of the priming relations (Experiments 2–5). To guide this investigation, we developed a process model of multiplicative reasoning with fractions to account for the types of equations for which a multiplicative schema will, or will not, potentially yield a priming effect. Computer code for the model is provided in the Supporting Information.

We assume that people may employ various strategies to evaluate the correctness of a simple multiplication equation. For equations with fractions, perhaps the most obvious strategy is to evaluate the equation in a manner similar to a whole-numbers division equation. For example, $3 \times 8/6 = 4$ could be evaluated by first multiplying 3×8 and then dividing that product by 6. This strategy will work on any fraction multiplication equation, even if the equation itself is false. However, a limitation of this strategy is that multiplying the whole number and the fraction numerator may lead to a very large product.

For equations in which common factors are available, people may solve multiplication equations involving fractions using a simplification strategy that minimizes the calculations required. One potential simplification strategy to evaluate $3 \times 8/6 = 4$ would be to simplify the fraction 8/6 to 4/3. The simplified equation is now $3 \times 4/3$, which reduces to 4. Another possible simplification strategy would be to reduce the whole number multiplier and the denominator of the fraction (essentially, $3 \times 1/6 = 1/2$), which would result in 8/2, and note that 8 divided by 2 is 4. One consequence of such simplification strategies is that common-factor relations between the whole number, fraction numerator, and fraction denominator are activated.

It is important to note that in the Complete-Match fraction equations used in Experiment 1, the relevant pairs of prime and target equations could always be solved by at least one of the simplification strategies, because common factors linked either the whole number and denominator, the numerator and denominator, or both. The false equations (half of the total set) could not be simplified in such a way (e.g., a false equation might be 4 × 5/3 = 10, which does not simplify to 10).

We propose that in the process of evaluating the prime equation (i.e., the first trial in a successive primed pair), participants will typically activate a common factor between the whole number and denominator, or the numerator and denominator, to simplify the equation (see Fig. 3, left panel). Hence, if the prime were 4 × 6/8 = 3, then the numbers 4, 6, 8, and 3 would provide the initial sources of activation. In Fig. 3, the arrows connecting numbers represent the relation "is a factor of." These arrows are depicted as unidirectional to reflect the asymmetry of the relational roles (factor and product), but we assume activation can spread in both directions (see Campbell, 1999a). We assume for simplicity that the two nearest factors and products of a number are activated (i.e., those numbers depicted in the example networks, which are the two successive multiples within a times table: 2 activates 4 and 8). (Additional factors/products may also be activated, but two are sufficient for the equations used in our experiments.) In solving the prime equation by simplification, relevant factor relations become highlighted, while irrelevant ones are deactivated. Thus, in solving the prime equation (bottom left in Fig. 3, the fact that 3 is a

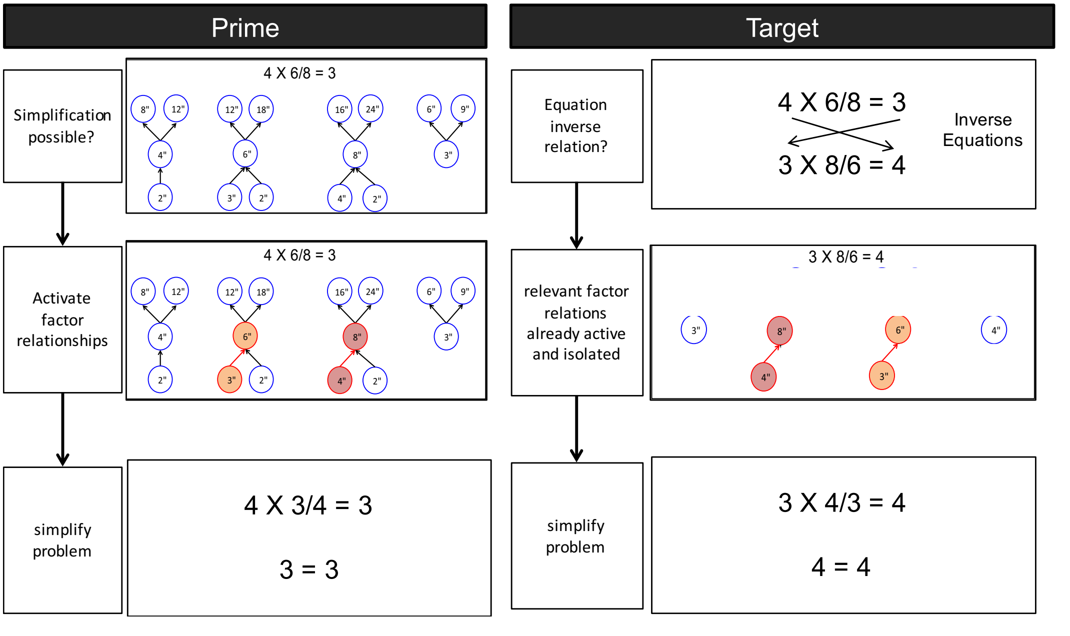

Fig. 3. Process model of priming between prime and target equations. The model requires that a prime and target equation must (a) share a structural equivalence and (b) activate the same common-factor relations across equations. When solving the prime equation, the solver may use a simplification strategy to reduce common factor relationships within the equation. Then, solving the target equation will yield a global speed-up in solving the prime because the equations satisfy the equation inverse relation and thus require the same common-factor relations for a solution.

factor of 6, and/or 4 is a factor of 8, would become highly active. If the target trial in a successive primed pair (Fig. 3, right panel) is the inverse equation 3 × 8/6 = 4, these same factor relations are precisely those necessary for simplification. Hence, the process of activating relevant factor/product relations will have a "head start" in the target equation, relative to the prime equation (leading to the prediction of parallel functions in a Vincentile analysis; see Fig. 2). Accordingly, priming is expected in that the target equation will be solved more quickly than the prime, where the order of the prime and target are counterbalanced to control for other sources of difficulty (i.e., an equation would appear as the prime for one participant but as the target for another).

This process model suggests that when a participant evaluates the target equation, two factors determine whether priming may be obtained. First, by definition, the connection between the prime and target trials depends on whether they share the equation inverse relation. However, in order for this relation to actually facilitate solution of the target trial, there must at least be some implicit recognition of how the two equations relate to one another. Importantly, in some cases recognizing the inverse structure may be all that is needed for priming to occur. In those cases, if the inversion is easily recognized, then no further calculation is necessary on the target trial.

The second factor that contributes to successful priming is that the same common-factor relations can be activated in the prime as in the target. This may be more important for cases with lower perceptual similarity, for which it will be more difficult to recognize the structural similarity. For example, in the prime trial shown in Fig. 3, the facts that 3 is a factor of 6 and/or 4 is a factor of 8 are activated in the evaluation of both the prime and target equations. If the fractions are at least related reciprocally (i.e., their simplified magnitudes are reciprocals), then priming may be obtained. However, the degree of similarity between reciprocals may vary with respect to how they are multiplicatively related (e.g., 1/4, 3/12, and 4/16, are all instances of 1/4, although the component parts of 3/12 and 4/16 do not share a divisible relationship). Perhaps highly expert mathematicians can easily represent these fractions in their simplest form (i.e., simplify both components with their greatest common factor), and hence might show priming for all reciprocals. However, our model assumes that people need to recognize common factors between the numerator and denominator of the fractions within the equations themselves, and do not simplify each to their simplest form. Accordingly, the model does not predict priming for equations that include reciprocals in forms that do not share a divisible relationship.

Table 2 provides a summary of the operation of the process model and its predictions. Importantly, the model is intended to explain the range and limits of relational priming with fractions for relatively expert solvers, who benefit from a well-established multiplicative schema. Less able solvers, presumably lacking a fully developed multiplicative schema, may show much more limited priming. For example, step 3 of the process model requires that solvers be fairly expert in their representation of multiplicatively related numbers. In cases where the numbers are not identical, a solver must be able to recognize that they share some multiplicative relation. A person with limited math skills might fail to perform the computation in step 3, in which case priming would be limited to cases in which the prime and target reciprocals are based on identical whole numbers. Experiment 2 was

Table 2

Basic logic of computer simulation of process model for priming of fraction problems via a multiplicative schema. The simulation tests whether or not priming is predicted for a given prime-target pair. The simulation correctly predicts priming for relationally related true primed fraction pairs, but not for true pairs that lack common factors in a consistent relational order, false primed pairs, or foil pairs. An outcome of 0 implies a prediction of no priming, an outcome of 1 implies a prediction of priming

Given a prime—target pair:
1. Check if a simplification strategy[a] is possible between the whole number and denominator, or the numerator and denominator, *within* each equation. This includes a check if the numbers are exact matches as in the Complete-Match true and the false primed equations.
If yes: continue to 2.
Else: return 0.
2. Check whether whole-number factor and answer have inverse positions across the prime and target equations (i.e., the whole-number factor in the prime should be the answer in the target and *vice versa*).
If yes, and fraction components are identical reciprocals: return 1.
If yes, and fraction components do not match identically: continue to 3.
Else: return 0.
3. Check whether the prime numerator and target denominator share common-factor relations (i.e. prime numerator is divisible by target denominator or *vice versa*), and whether the prime denominator and target numerator share common-factor relations (i.e. prime denominator is divisible by target numerator or *vice versa*).
If yes: return 1.
Else: return 0.

Notes. [a]A simplification strategy involves "canceling out" common factors between factors and the division in the equation. For fraction equations of the type included in our stimuli, there are two simplification strategies possible. One is to simplify the whole number multiplier and the denominator. The second is to simplify the numerator and denominator of the fraction. Our process model implies that the process of carrying out the simplification activates common factor relationships within an equation.

performed to assess the role of math expertise as a possible prerequisite for priming when the numbers composing the prime and target are not perceptually identical.

4. Experiment 2

Experiment 2 was designed to test how the relational priming between two inverse fraction equations differs as a function of perceptual and relational similarity and overall math knowledge or expertise. Participants were recruited from two universities that enroll students who vary widely in overall math knowledge. We varied the degree of perceptual similarity between inverse fraction equations. In the Complete-Match condition (used in Experiment 1), the four digits in the consecutive pairs of prime and target equations were identical and, therefore, the relationship between the fraction equations was perceptually salient (e.g., $3 \times 4/3 = 4$; $4 \times 3/4 = 3$). In the Between-Equation (B-E) Mismatch condition, only the whole numbers in the consecutive equations were the same (see Table 1). The fraction multipliers used different digits that maintained the equations' relational similarity (e.g., 3 × 4/3 = 4; 4 × 6/8 = 3, where 4/3 and 6/8 are reciprocals of one another), so that our process model predicts that relational priming is possible *if* the solver has a well-established multiplicative schema. We hypothesized that low-performing participants may show facilitation in the Complete-Match case, for which a perceptual strategy supports relational similarity, but not in the B-E Mismatch case, for which reliance on perceptual similarity is not possible. In terms of our process model, execution of step 3 (required when no perceptual strategy is available, as in the B-E mismatch case) may vary depending on the expertise of the participant.

4.1. Method

4.1.1. Participants

A total of 75 undergraduates participated in the study for course credit. Thirty-eight participants were undergraduates from California State University, Los Angeles (CSULA) (20 females), and 37 were undergraduates from UCLA (29 females).

4.1.2. Design, materials, and procedure

Participants completed a multiplication task that was identical in general format to that used in Experiment 1. However, Experiment 2 included an additional between-subjects condition: B-E Mismatch Fractions (see Table 1). In this condition, true primed pairs consisted of inverse equations related by common factors (3 × 4/3 = 4; 4 × 6/8 = 3) where 4 is a factor of 8 and 3 is a factor of 6. Therefore, while the numbers are no longer perceptually similar between equations, the inverse relation is maintained because one of the equations can be simplified (6/8 = 3/4). Participants were randomly assigned to one of two between-subjects conditions: Complete-Match fractions (N = 37) and B-E Mismatch fractions (N = 38). This task took between about 10–20 min depending on the participants' general math knowledge.

After performing the speeded multiplication task, participants completed an explicit assessment of general math knowledge, which was used to split the participants into relatively low- and high-performing groups. This math-assessment test involved a total of 25 multiple-choice problems, and eight problems requiring a solution (either equations or word problems). The test comprised three subsections, each designed to assess a domain of mathematical understanding: algebra, fractions, and multiplicative understanding (see Appendix S2). The algebra questions (adapted from DeWolf, Bassok, & Holyoak, 2015b) included basic equation solving questions, word problems, and evaluations of algebraic expressions. Fraction problems queried participants about equivalent fractions, ratio relationships, and the relation between the size of the numerator and denominator. Multiplicative questions asked about the greatest common factor of two numbers, the reciprocals of certain numbers, and lowest common multiples of two numbers. These problems were adapted from released questions on the 2008 California State Standards exam for Algebra I. Successful performance on this test would only require a level of understanding corresponding to basic high school math.

The math-assessment test was administered with paper and pencil. Participants were randomly assigned to one of three different random orders. They were encouraged to use space on the page to write out their work, and they were told not to use a calculator. This task took about 15 min to complete.

4.2. *Results and discussion*

4.2.1. *Math-assessment test*

Because the math-assessment test consisted of multiple-choice questions or questions for which there was only one correct answer, questions were scored on a 0, 1 basis. Final scores for the three problem subtests (Algebra, Fractions, and Multiplicative Understanding) were averaged across questions in a relevant subtest. The math assessment was found to be highly reliable (25 items: $\propto$ = .82).

Mean overall accuracy, across all of the problem subsets, was 76% (*SD* = 17; minimum score = 36%, maximum score = 100%). For the algebra subtest, overall accuracy was 75% (*SD* = 18, minimum score = 41%, maximum score = 100%). For the fraction subtest, overall accuracy was 73% (*SD* = 26, minimum score = 13%, maximum score = 100%). Finally, for the multiplicative-understanding subtest, overall accuracy was 81% (*SD* = 15, minimum score = 50%, maximum score = 100%). The participants thus spanned a broad range in overall math ability, enabling us to examine differences in performance between low- and high-performing participants. Importantly, we define low- versus high-performing relative to the sample that we collected and their performance on the general math test, using a median split. "High performing" participants were defined as those scoring highest relative to our sample. Of course, this group does not encompass all levels of math ability—we did not collect data from high-level experts (e.g., math PhDs); nor did we capture very low levels of math performance (as all of our participants were enrolled in college). On average, mean accuracy was 88% for UCLA students and 65% for CSULA students; however, the ranges of scores from UCLA and CSULA overlapped, with low- and high-performing participants coming from both schools (86% of low-performing students came from CSULA and 82% of high-performing students came from UCLA).

4.2.2. *Multiplication priming performance and math performance level*

Fig. 4 shows the mean accuracy for prime and target equations separately for the low- and high-performance groups (determined by the median split). A 2 (math performance level: high vs. low) × 2 (equation type: prime vs. target) × 2 (fraction equation type: Complete-Match vs. B-E mismatch) ANOVA analysis revealed a reliable effect of math performance level ($F(1, 71) = 11.26$, $MSE = 3.2$, $p = .001$). No other effects on accuracy were reliable.

A parallel 2 (math performance level: high vs. low) × 2 (equation type: prime vs. target) × 2 (fraction equation type: Complete-Match vs. B-E mismatch) ANOVA was conducted for RTs. A significant three-way interaction was obtained ($F(1, 71) = 11.20$, $MSE = 13.90$, $p = .001$), indicating a differential priming effect for low- and

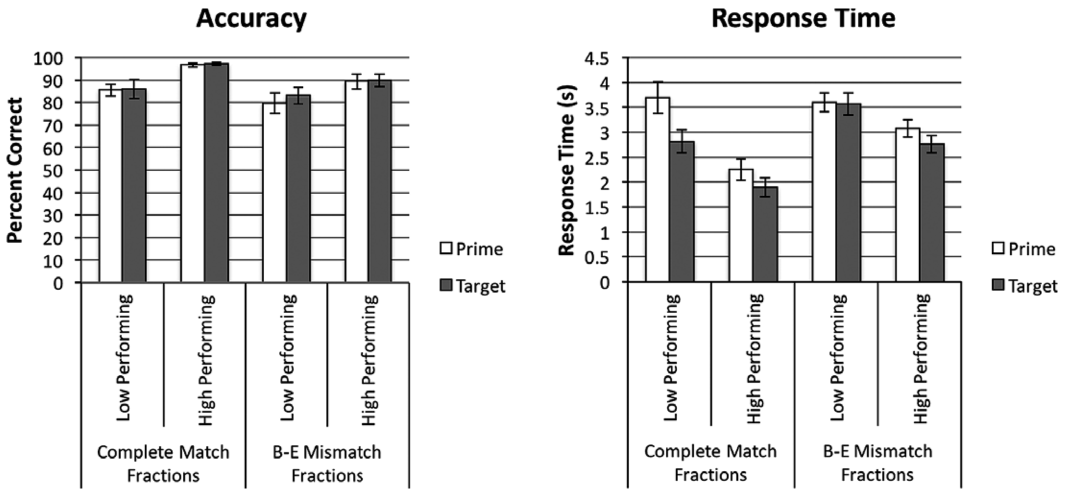

Fig. 4. Mean accuracy and response time for Complete-Match and B-E Mismatch conditions, for high- and low-performing participants on true prime and target equations (Experiment 2).

high-performers depending on the type of fraction equations. For Complete-Match equations, there was a significant two-way interaction between equation type and math performance level ($F(1, 35) = 5.65$, $MSE = 22.68$, $p = .02$), indicating that the priming effect varied depending on math performance level. Significant priming effects were found for both the high-performing group (prime: 2.25 s vs. target: 1.89 s; $F(1, 35) = 5.58$, $MSE = 45.35$, $p = .02$) and the low-performing group (prime: 3.70 s vs. target: 2.81 s; $F(1, 35) = 29.3$, $MSE = 45.35$, $p < .001$).

The mean speed-up from prime to target (prime RT − target RT) was calculated for each participant in each performance group. The low-performing group showed a significantly greater speed-up compared to the high-performing group (.73 s vs. .23 s, $t(24) = 3.18$, $p = .004$). Thus, when pairs of mathematical equations were perceptually similar (Complete-Match condition), the priming effect held for both high- and low-performing students and was actually larger for low-performing students. This priming difference observed in the Complete-Match fraction condition may in part be related to general differences in RTs between the low- and high-performing groups. Mean prime and target RTs for the low-performing group were considerably slower than for the high-performing group. Thus, although the priming speed-up for the low-performing group was much larger than that for the high-performing group, these differential gains may be related to the longer overall RTs of the former group.

For the B-E mismatch equations, there was also a significant two-way interaction between equation type and math performance level ($F(1, 36) = 7.35$, $MSE = 53.64$, $p = .01$). A significant priming effect was again found for the high-performing group (prime: 3.08 s vs. 2.76 s; $F(1, 36) = 18.75$, $MSE = 10.72$, $p < .001$), but not for the low-performing group (prime: 3.61 s vs. target: 3.57 s; $F(1, 36) = .25$, $MSE = 10.72$, $p = .62$). The mean speed-up was calculated for each participant. For the B-E Mismatch fraction condition the high-performing group showed a significantly greater average speed-up than did the low-performing group, whereas the latter group did not show a priming effect at all (.27 s vs. −.06 s, $t(27) = 2.33$, $p = .03$).

4.2.3. Analysis of response time distributions

In a manner similar to Experiment 1, we conducted a Vincentile analysis to analyze the priming effect across the distribution of RTs. The Vincentile analysis revealed that target equations were overall faster than prime equations for the Complete-Match fraction condition, for both low performers (prime: 3.39 s vs. target: 2.63 s; $F(1, 16) = 13.25$; $MSE = .62$; $p = .002$) and for high performers (prime: 2.15 s vs. target: 1.80 s; $F(1, 19) = 8.43$; $MSE = .72$; $p = .009$), with no reliable interaction between bin number and prime versus target for either low performers ($F(5, 80) = 1.24$; $MSE = .10$; $p = .35$) or high performers ($F(5, 95) = 1.49$; $MSE = .18$; $p = .11$).

For the B-E Mismatch condition, the Vincentile analysis (like the main analysis reported above) showed no evidence of a priming effect for low performers (prime: 3.68 s vs. target: 3.57 s, $F(1, 18) = 1.011$, $MSE = .67$ $p = .33$), and no interaction between bin number and prime versus target ($F(5, 90) = .142$ $MSE = 2.97$, $p = .98$). In contrast, for high performers a significant priming effect with B-E Mismatch pairs was observed (prime: 3.04 s vs. target: 2.72 s, $F(1, 18) = 14.34$, $MSE = .60$, $p = .001$), with no evidence of an interaction between bin number and prime versus target ($F(5, 90) = 2.69$, $MSE = 1.14$, $p = .08$). The Vincentile analysis thus indicates that the priming obtained for participants at both ability levels with Complete-Match pairs, and for high-performing participants with B-E Mismatch pairs, was consistent across the full distribution of RTs, rather than being specific to some selective range, such as the slowest responses (i.e., most difficult equations).

5. Experiment 3A

In Experiment 2, high-performing math students demonstrated relational priming even on target equations that were only relationally related to the prime (i.e., not based on identical component whole numbers). The remaining experiments focus on the performance of students (drawn from the same general population as the majority of the high-performing students in Experiment 2) across a broader range of perceptually dissimilar but relationally related pairs. In Experiments 3A and 3B we examined whether perceptual and relational changes in format modulate the priming effect for fraction-format equations. The goal of these experiments is to test whether the priming effect we observed with high-performing participants in Experiment 2 is simply some perceptual artifact or reflects the ability of well-educated adults to exploit the relational structure of fractions.

Experiment 3A tested the limits of the priming effect found for the fraction multiplication equations by varying the order of the numbers in the equations within a primed pair. The B-E Mismatch pairs in Experiment 2 showed a priming effect despite a perceptual mismatch between equations, which suggests the priming is deeper than just a perceptual match. However, in Experiments 1 and 2, fraction equations were always presented in the format: *Whole Number* × *Fraction* = *Whole Number*. Therefore, one could argue that the priming effect might be due to some procedural fluency for the inverse-related equations because of the ease of applying the same strategy in the same order across the two equations. To assess this possibility, in Experiment 3A half of the equations were instead presented in the format: *Fraction* × *Whole Number* = *Whole Number*. If priming depends on the superficial match between order of presented number types, then it will be reduced or eliminated when the prime and target have different number orderings (e.g., *WN* × *F* = *WN*; *F* × *WN* = *WN*). However, if priming relies on detecting the inverse structure across equations (as our process model predicts), then priming will still be obtained despite variations in number order because the fraction serves to parse the equation and thereby highlight the relevant factor relations.

5.1. Method

5.1.1. Participants

A total of 31 UCLA undergraduates (M_{age} = 19.7; females = 20) participated in the study for course credit.

5.1.2. Design, materials, and procedure

The design, materials, and procedure were the same as in previous experiments except that the fraction-format stimuli for Experiment 3A were in a form we term the Within-Equation (W-E) Mismatch condition (see Table 1). In contrast to Complete-Match pairs, these stimuli involved individual equations in which the components of the fraction (numerator and denominator) were not identical to either the whole-number multiplicand or the whole-number product (i.e., there was a mismatch between the fraction and the whole numbers). For example, a W-E Mismatch pair might be 4 × 3/2 = 6 followed by 6 × 2/3 = 4. In the first equation (prime), the whole numbers (4 and 6) do not match either of the components of the fraction (3 and 2); the same is true for the second equation (target). Note that in contrast to the B-E Mismatch condition tested in Experiment 2 (e.g. 3 × 4/3 = 4; 4 × 6/8 = 3), the W-E Mismatch condition involves the absence of identity for numbers *within* each of the two successive equations.

The W-E Mismatch fraction equations were created by either multiplying the fractions from the Complete-Match condition by 2/2 or 3/3, or by reducing those fractions by the same factors. False prime pairs and foil pairs were equivalent to those used in Experiment 1 (rewritten in the division format for that condition).

There was one within-subjects factor based on B-E Mismatch fractions: order (consistent, inconsistent). The consistent-order condition consisted of true primed pairs that had the same whole number-fraction order (e.g., *WN* × *F* = *WN*; *WN* × *F* = *WN*). The inconsistent-order condition was composed of true primed pairs that had the opposite ordering of the whole number and fraction in prime and target equations (e.g., *WN* × *F* = *WN; F* × *WN* = *WN*). In the inconsistent-order condition, either the prime or the target could have the whole number first. Half of the true primed pairs (15) had a consistent ordering and the other half had an inconsistent ordering. To ensure that the variation in order between the whole number and fraction was present for all equation types (and not just the true primed pairs), the foils and false primed pairs also were adjusted such that half of these pairs also had an inconsistent ordering.

5.2. Results and discussion

5.2.1. Accuracy

Across all equations, mean accuracy was similar to that found in previous experiments (86%). Fig. 5 (left) shows the mean accuracy values for the true primed prime and target equations separated by consistent versus inconsistent number orderings within pairs. A repeated-measures ANOVA with two factors (order: consistent vs. inconsistent) × (prime vs. target) revealed a reliable overall increase in accuracy from prime to target equations (prime: 82% vs. target: 85%, $F(1, 30) = 6.69$, $MSE = .005$, $p = .02$). There was no significant interaction between order and prime versus target ($F(1, 30) = .16$, $MSE = .006$, $p = .69$), indicating that the magnitude of the priming effect did not vary depending on whether or not the order of the whole number and fraction was consistent across paired equations.

5.2.2. Response times

Across all equations, mean RTs for correct trials were similar to those found in previous experiments (3.50 s). Fig. 5 (right) shows the mean RTs for true prime and target equations as a function of consistent versus inconsistent number ordering. A repeated-measures ANOVA with two factors (order: consistent vs. inconsistent) × (prime vs. target) revealed a significant overall speed-up in RTs from prime to target equations (prime: 3.32 s vs. target: 2.95 s, $F(1, 30) = 25.80$, $MSE = 1.39$, $p < .001$), indicating a priming effect. There was no significant interaction between order and prime versus target ($F(1, 30) = 2.19$, $MSE = 1.32$, $p = .15$), again indicating that the magnitude of the priming effect did not vary depending on whether there was a match between the order of the whole number and fraction.

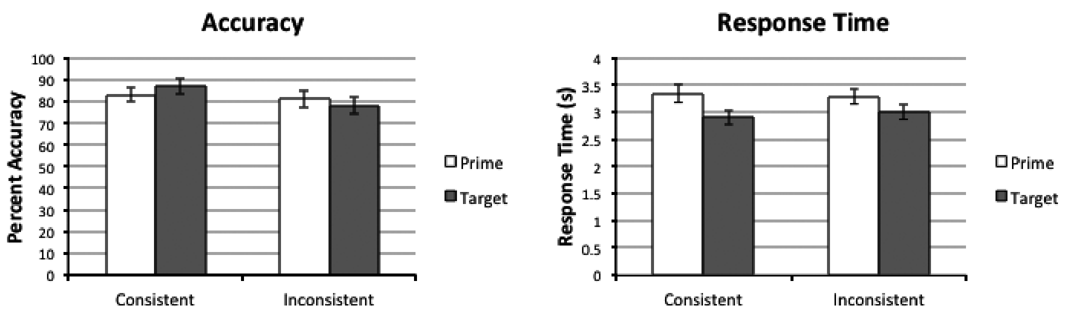

Fig. 5. Mean accuracy and response times for true B-E fraction prime-target pairs with consistent versus inconsistent number orderings (Experiment 3A).

6. Experiment 3B

Experiment 3A demonstrated that varying the order of the whole number and fraction between prime and target equations did not reduce the magnitude of relational priming between inverses. This finding is consistent with our schema-based process model, which attributes priming to activation of shared relations and factors. Priming occurs despite superficial changes in formatting that nonetheless preserve the grouping of numbers involved in fraction and whole-number relationships.

Experiment 3B tests the complementary prediction: Formatting changes that break up relevant groupings will reduce or eliminate priming. In contrast to the formatting variation tested in Experiment 3A, the variation investigated in Experiment 3B breaks the equation inverse relation (as specified in our model) between the prime and target equations. We assessed whether preserving the relational components across inverse equations is necessary for priming. Our process model assumes that priming is based on the inverse relation between equations (e.g., $3 \times 4/3 = 4$ is the inverse of $4 \times 3/4 = 3$). However, the priming effect might instead be due to seeing the same numbers repeated from one equation to the next, rather than specifically requiring an inverse relation between the two equations. Consider the example pair, prime: $3 \times 4/3 = 4$ and target: $4 \times 3/4 = 3$. An alternative potential target, which includes the same numbers, is $3 \times 4/4 = 3$. In this alternative target, the inverse structure between the two equations is no longer preserved. Although the numbers are the same across the two equations, the structure of the equations affords two different kinds of strategies (in the second target, 4/4 can easily be simplified to 1). Accordingly, our process model predicts that priming will *not* be obtained for the alternative target.

Experiment 3B included prime-target pairs in which there is no inverse relationship within equation pairs. For example, an example of a standard W-E Mismatch pair is $12 \times 4/6 = 8$; $8 \times 6/4 = 12$ (where 4/6 and 6/4 are reciprocals and 12 and 8 swap places inversely). In contrast, Experiment 3B introduced cases in which a whole number and numerator are interchanged between the prime and target, as in $12 \times 4/6 = 8$; $6 \times 8/4 = 12$. Although the paired equations are both true, and are composed of the same integers, reciprocal and inverse relations no longer hold between them. That is, 4/6 and 8/4 are not reciprocals and thus there is no pair of whole numbers that swap places. Although the same common-factor relations might be activated in the two equations, the fact that the fractions are unrelated obscures the mapping between numbers that could be factored across the prime and target.

6.1. Method

6.1.1. Participants

A total of 29 UCLA undergraduates ($M_{\text{age}} = 20.3$, females = 18) participated in the study for course credit.

6.1.2. Design, materials, and procedure

The design, materials, and procedure were closely modeled on those used in Experiment 3A. There was one within-subjects factor, based on pairs generated from the W-E Mismatch pairs used in Experiment 3A. Numbers in half of the true primed pairs were in the standard relational order (e.g., 12 × 4/6 = 8; 8 × 6/4 = 8), and in the other half they were in a non-relational order in which a whole number and numerator in the prime equation were swapped to form the target (e.g., 12 × 4/6 = 8; 6 × 8/4 = 12). Analogous adjustments were made to the foil equations and false primed pairs such that on half of all the equations, the whole number and numerator were interchanged relative to their traditional counterparts.

6.2. Results and discussion

6.2.1. Accuracy

Across all equations, mean accuracy was similar to that observed in previous experiments (87%). Fig. 6 (left) shows the mean accuracy values for true prime and target pairs, separated by relational versus non-relational orders. A repeated-measures ANOVA with two factors (format: relational vs. non-relational) × (prime vs. target) revealed no priming effect in accuracy across all equations (prime: 80% vs. target: 83%, $F(1, 28) = 3.73$, $MSE = .009$, $p = .06$), and no reliable interaction between format and prime versus target ($F(1, 28) = .21$, $MSE = .009$, $p = .65$). There was a significant effect of format such that equations with a relational order yielded significantly higher accuracy than those with a non-relational order (85% vs. 78%, $F(1, 28) = 20.62$, $MSE = .006$, $p < .001$).

6.2.2. Response times

Across all equations, mean RTs for correct trials were similar to those obtained in previous experiments (3.11 s). Fig. 6 (right) shows the mean RTs for prime and target equations, separated by relational and non-relational order. A repeated-measures ANOVA with two factors (format: relational vs. non-relational order) versus (prime vs. target) revealed

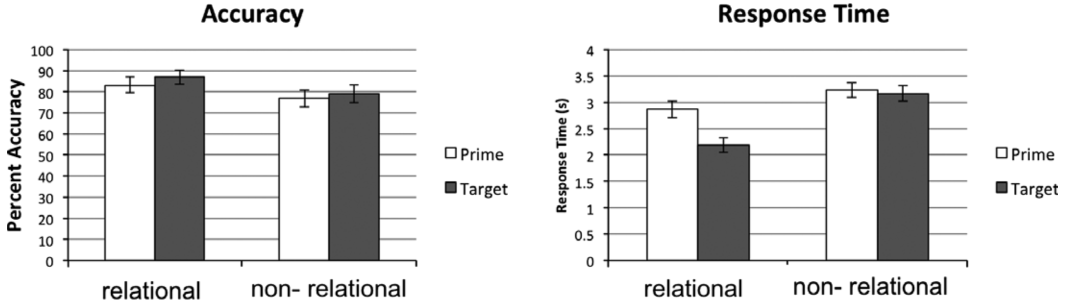

Fig. 6. Mean accuracy and response times for true B-E fraction prime-target pairs with relational versus non-relational order (Experiment 3B).

a significant interaction ($F(1, 28) = 5.11$, $MSE = .93$, $p = .03$), indicating that there was a difference in the priming effect depending on the format of the equations. Planned comparisons showed that a significant priming effect was obtained for the standard pairs with relational orders (prime: 3.20 s vs. target: 2.87 s: $F(1, 28) = 12.13$, $MSE = 2.39$, $p = .002$). However, no reliable priming was observed for pairs with the non-relational order (prime: 3.23 s vs. target: 3.17 s: $F(1, 28) = .36$, $MSE = 2.89$, $p = .55$).

The results of Experiment 3B indicate that mathematical variations that alter the structure of the prime and target so as to eliminate reciprocal and inverse relations diminish, and perhaps eliminate, the priming effect. Whereas format changes that preserve the fraction chunk (order of numbers) do not affect the priming effect (Experiment 3A), changes that break up that fraction chunk (swapping a whole number with the numerator of the fraction in the target) impair relational priming (Experiment 3B).

7. Experiment 4

The results of Experiments 3A and 3B indicate that the fraction notation can chunk an equation in a way that highlights the inverse relation between the prime and target. In our stimuli, four numbers, each in a specific role, are being switched between prime and target: (1) Whole Number × (2) Numerator/(3) Denominator = (4) Whole-Number Product. The fraction highlights the "slots" for the numerator and denominator, which reverse their positions across the prime and target. In Experiment 3A, we found that the position of the fraction relative to the whole-number factor is irrelevant. However, in Experiment 3B, we found that varying the elements within the fraction and whole number so that relational chunks are no longer preserved across the inverse equation pairs eliminates the priming effect. The fraction format itself seems to help preserve the relational chunks across the prime and target problems. However, it is still possible that adults manipulate the components within the fraction separately. In Experiment 4, we assessed whether solvers indeed treat the fraction as two separate component pieces, rather than a single holistic unit with a single integrated magnitude, by further varying the perceptual features of the equation.

To achieve our aim, Experiment 4 introduced equations formatted either as fractions (as in Experiments 1–3) or in a division format (see Fig. 7). Fraction-format equations,

Fraction Format	Division Format	
	Relational Order	Non-relational Order
3 x 8/6 = 4	$\frac{3 \times 8}{6} = 4$	$\frac{8 \times 3}{6} = 4$
4 x 6/8 = 3	$\frac{4 \times 6}{8} = 3$	$\frac{4 \times 6}{8} = 3$

Fig. 7. Examples of the different formats used in the fraction and division conditions in Experiment 4.

which maintained their relational ordering, served as a control condition. The division-format condition included two subconditions: relational and non-relational order equations (see Fig. 7). The relational division pairs, like the fraction pairs, highlight the shared structure between prime and target, and hence are expected to yield priming. In contrast, the non-relational division pairs, like the non-relational fraction pairs used in Experiment 3B, obscure the shared structure and hence are not expected to yield priming. If participants treat fractions as components of an overall division equation, then performance on the relational division and fraction pairs should be comparable. However, if participants treat fractions as a single holistic unit with a single rational-number magnitude, then performance on the fraction equations should not resemble that on the division equations.

7.1. Method

7.1.1. Participants

A total of 74 UCLA undergraduates (M_{age} = 21.2; females = 59) participated in the study for course credit.

7.1.2. Design, materials, and procedure

The design, materials, and procedure were similar to those used in Experiment 1. There were two between-subject conditions: fraction format versus division format. The fraction format included stimuli in a form of W-E Mismatch pairs (see Table 1).

The same basic equations were also used in the division-format condition, except that the equations appeared in a division format without an explicit fraction multiplier (see Fig. 7). There were two subtypes of division equations given within subjects. The first (relational order) were ones that preserved the relational ordering of the fraction condition. For example, in Fig. 7, the numbers comprising the reciprocal (8/6 and 6/8) maintain their inverse positions across the two equations. The other order (non-relational) shown in Fig. 7 has 3/6 and 6/8 in the inverse position, losing the place mapping across the two equations. Such a difference in ordering may encourage participants to use different simplification strategies across the two equations, thus limiting the priming effect.

7.2. Results and discussion

7.2.1. Accuracy

Fig. 8 (left) shows mean accuracy levels for prime and target equations, for the fraction condition and the two division conditions (relational vs. non-relational order). Because the fraction and division conditions did not form a factorial design, we analyzed data for each format separately. There was no evidence of priming based on the accuracy measure, as there was no main effect of trial type ($F(1, 72) = 3.89$, $p = .06$) and no interaction between trial type and format condition ($F(1, 72) = .32$; $p = .57$). Across all trials, there was no difference in accuracy between the fraction condition and the division condition (90% vs. 90%; $t(72) = .007$; $p = .994$).

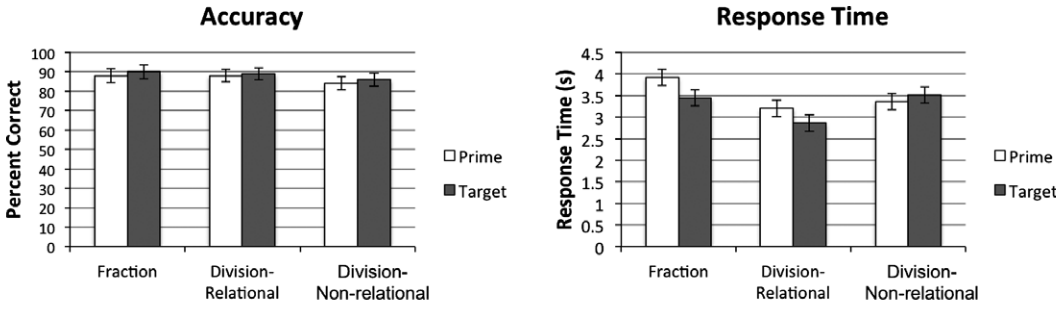

Fig. 8. Mean accuracy and response times for true-prime prime and target equations for fraction and division formats (Experiment 4).

7.2.2. Response times

Fig. 8 (right) shows the mean RTs for prime and target equations by format condition. A significant priming effect was observed for the fraction-format equations (prime: 3.92 vs. target: 3.45, $t(36) = 3.01$, $p = .005$). A separate repeated-measures ANOVA with 2 factors (prime vs. target) × (Format: relational vs. non-relational order) was used to assess whether priming effects for division equations differed as function of the relational ordering of the numbers. This analysis revealed a significant interaction ($F(1, 36) = 4.17$, $MSE = 3.47$, $p = .048$). For the relational-order pairs, a significant priming effect was obtained (Prime: 3.20 s vs. Target: 2.86 s, $F(1, 36) = 6.55$, $p = .02$). In contrast, there was no priming effect for pairs with the non-relational order (Prime: 3.36 s vs. Target: 3.51 s, $F(1, 36) = .43$, $MSE = 1.95$, $p = .51$). This finding indicates that priming occurs when the relational correspondence of factors is preserved across equations in division format, but when this relational correspondence is broken, different common factors are activated in the paired equations, resulting in different solution strategies and hence no priming.

Across all equations (including foils), and collapsing across the two formats for division equations, participants responded more slowly to fraction-format equations than to division-format equations (3.57 s vs. 2.79 s; $t(72) = 3.65$, $p = .001$). Although the division format (non-relational order) did not yield facilitation between inverse equations (which depends on recognizing relational similarities *across* equations), it may have facilitated activation of common factors *within* each individual equation. This possibility is consistent with previous work showing that different visual representations of arithmetic equations can differentially highlight the elements that bind across operations (e.g., Kirshner, 1989; Landy, Brookes & Smout, 2014; Landy & Goldstone, 2007b, 2010). In Experiment 4 case, the division format made it easier to simplify either one of the numerator numbers with the denominator.

As in the previous experiments, we performed a Vincentile analysis to assess whether the priming effect found for the fraction format and relational-order division format was uniform or variable across bins of response item RTs. The pattern of priming obtained in Experiment 1 (see Fig. 2) was observed for the fraction format and relational-order division equations. There was a general speed-up in RTs from prime to target both for fraction equations (prime: 3.90 s vs. target: 3.36 s; $F(1, 36) = 8.95$, $p = .005$), and for relational-order division equations (prime: 3.21 s vs. target: 2.86 s $F(1, 36) = 4.20$, $MSE = .16$, $p = .04$). Neither type of equation yielded any evidence of an interaction between prime versus target and bin number (fractions: $F(5, 180) = 1.55$, $MSE = .84$, $p = .176$; relational-order division: $F(5,180) = .055$, $MSE = 2.43$, $p = .99$). For division equations in the non-relational order there was no evidence of priming (prime: 3.35 s vs. target: 3.20 s; $F(1, 36) = .75$, $MSE = 3.31$, $p = .39$) nor of any interaction with bin number ($F(5, 180) = .166$, $MSE = .67$, $p = .98$) for the non-relational order division equations.

Overall, the results of Experiment 4 reveal that for participants with relatively high math ability, priming between inverse equations was consistently found with fraction-format equations that preserve the relational correspondence between equations. For the division format, priming was only observed for the relational order, which (like fractions) promotes activation of similar common factor pairs across equations, yielding a speed-up in solution times. These results closely mirror those of Experiment 3B, suggesting that, as long as the equation inverse correspondence across equations is maintained, a priming effect is observed.

8. Experiment 5

Thus far, our results have shown that equations that are relational inverses of one another yield a significant priming effect (Experiments 1 and 2), whereas equations that contain the same numbers but lack the relational correspondence do not (Experiments 3A, 3B, and 4). We also found that participants recognize relational inverse equations that have common factors (Experiment 3). The goal of Experiment 5 was to further test the limit of the relational priming effect by including reciprocal fractions that match in magnitude but do not contain fraction components divisible by one another. The stimuli that we used in Experiment 5 combined the features of W-E Mismatch equations (Experiment 3A and 3B) and B-E Mismatch equations (Experiment 2). In these *Double-Mismatch* fraction equations, the components of the fraction did not match the whole numbers in the same equation, nor did they match the components of the fraction in the paired equation (e.g., $8 \times 6/4 = 12$, followed by $12 \times 2/3 = 8$). For such pairs, the relevant fraction components can be simplified to match across the prime and target.

The Double-Mismatch equations comprised two subsets. The first were problems for which the numbers within the fraction reciprocal pairs were divisible by one another (Double-Mismatch-D). For example, in the pair $8 \times 6/4 = 12$; $12 \times 2/3 = 8$, 3 is divisible by 6 and 2 is divisible by 4. We also included a variety of Double-Mismatch equations in which the fractions are reciprocals of one another but the numerator and denominator of the prime and target pairs are *not* divisible by each other (Double-Mismatch-ND). Consider $3 \times 24/9 = 8$ and $8 \times 6/16 = 3$, where 24/9 and 6/16 are reciprocals of one another. Although 24/9 and 16/6 are equivalent in magnitude, 24 cannot be simplified to 16 and 9 cannot be simplified to 6. For such Double-Mismatch, ND equations, different hypotheses about processing lead to different predictions about priming. Our process model assumes that if the two sets of numbers activate the same set of factors or multiples (and are viewed as multiplicatively related), then there should be priming. For example, a highly expert mathematician (perhaps beyond the level of the UCLA undergraduates that we recruited for this study) may automatically note that 24/9 and 6/16 are reciprocals (as both are forms of 8/3 and 3/8). It is also possible that for every equation participants always reduce fractions to their simplest form (generating 8/3 and 3/8), thus creating the basis for priming. However, if priming is based on activating common factors between the two equations (and not noting reciprocal magnitudes), then priming should be observed for Double-Mismatch-D pairs but not for Double-Mismatch-ND pairs.

8.1. Method

8.1.1. Participants

Participants were 74 UCLA undergraduates (M_{age} = 20; 48 females) who received course credit. Thirty-seven participants were randomly assigned to each of the two between-subjects conditions.

8.1.2. Design, materials, and procedure

The design, materials, and procedure were similar to those used in Experiment 1. We included the Complete-Match fraction condition to compare with the novel fraction conditions introduced in Experiment 5. A new between-subjects condition was used to test our hypothesis concerning the critical role of common-factor relations. This condition included two new types of true inverse equations. In Double-Mismatch fraction equations (e.g., $8 \times 6/4 = 12$ followed by $12 \times 2/3 = 8$), the fractions (6/4 and 2/3) do not have components that match either of the whole numbers (8, 12), nor do they share any common components with each other. There were 21 Double-Mismatch-D equations (only a limited number of such equations that can be generated with this criterion without the numbers becoming too large). An additional nine true-primed pairs were Double-Mismatch-ND. For example, one such pair is $2 \times 21/6 = 7$ followed by $7 \times 4/14 = 2$. In this pair, 21/6 and 14/4 are magnitude-equivalent fractions (hence, 21/6 and 4/14 are reciprocals of one another); however, 21 is not divisible by 14 and 6 is not divisible by 4. This condition thus differed from all of the previous types of fraction pairs, in which the numbers were divisible (e.g., 6/4 and 3/2; 6/8 and 3/4). The rest of the equations in the experiment (foils and false primed equations) were identical to those included in Experiment 1.

8.2. Results

8.2.1. Accuracy

As in the previous experiments, accuracy for the true primed equations was averaged for each participant. Mean accuracy values for each equation of the primed pairs for the three true prime conditions are shown in Fig. 9 (left). Because the Double-Mismatch-D versus Double-Mismatch-ND manipulation was conducted within-subjects, this analysis was performed separately from the overall ANOVA, comparing the two between-subjects conditions. A 2 (fraction type) × 2 (prime vs. target) mixed = factors ANOVA did not yield a significant interaction ($F(1, 72) = .62$, $MSE = .002$, $p = .44$), nor a significant main effect of priming ($F(1, 72) = .02$; $MSE = .002$; $p = .88$), indicating that as in all the previous experiments (except Experiment 3A), there was no priming effect for accuracy.

8.2.2. Response times

There was an overall difference in RTs between the two between-subjects equation types ($F(1, 72) = 7.63$, $MSE = 1.47$, $p = .007$). Planned comparisons showed that the Double-Mismatch fractions were slower than the Complete-Match fractions (3.83 s vs. 3.05 s; $t(72) = 2.76$; $p = .007$).

Fig. 9 (right) shows the RTs for true primed pairs by fraction type. A 2 (fraction type) × 2 (prime vs. target) mixed = factors ANOVA did not yield a significant interaction between fraction type and prime versus target ($F(1, 72) = .05$; $MSE = .79$; $p = .82$). There was a significant overall priming effect for both the Complete-Match and Double-Mismatch-D conditions (prime: 3.95 s vs. target: 3.39 s; $F(1, 72) = 11.45$, $MSE = .79$, $p < .001$).

We also tested the Double-Mismatch-ND condition separately and found that there was no priming effect for these equations (prime: 5.56 s vs. target: 5.94 s; $t(36) = .83$, $p = .41$). This result indicates that there is a limit on how far the priming effect extends for true inverse fraction equations. Specifically, the prime fraction needs to share common factors with the fraction in the primed equation, rather than simply a reciprocal relation to linking the two fractions to one another.

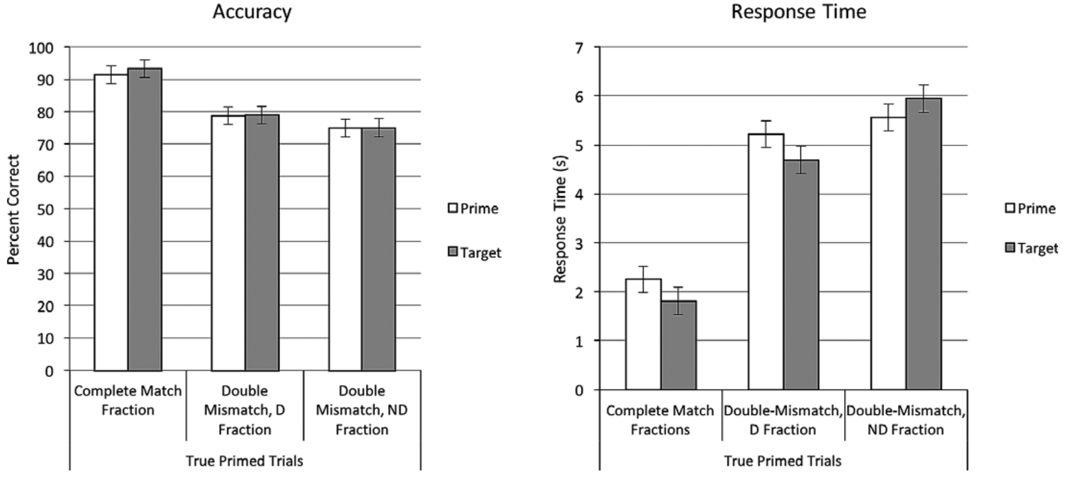

Fig. 9. Mean accuracy and response times for primed equations in Complete-Match, Double-Mismatch-D, and Double-Mismatch-ND conditions (Experiment 5).

In addition, we assessed whether a priming effect was obtained for the false primed equations. Although there was a significant interaction between prime versus target and number type ($F(1, 72) = 5.47$, $MSE = .54$, $p = .02$), this effect was due to significantly *slower* RTs for target equations in the Double-Mismatch condition (prime: 4.00 s vs. target: 4.45 s; $t(36) = 2.56$, $p = .02$). There was no priming effect for false prime Complete-Match pairs (prime: 3.66 s vs. target: 3.54 s; $t(36) = .69$, $p = .49$).

As in the previous experiments, we performed a Vincentile analysis to assess whether the priming effect found for the Double-Mismatch fractions was similar in nature to the priming effect found for other relationally similar fraction pairs. The pattern of general speed-up observed previously (Experiment 1) for the Complete-Match condition was replicated. In addition, for Double-Mismatch-D fractions we also found a general speed-up in RTs from prime to target (prime: 4.96 s vs. target: 4.58 s; $F(1, 36) = 4.47$; $MSE = .33$, $p = .04$), with no reliable interaction between prime versus target and bin number ($F(5, 180) = 2.43$, $MSE = 1.06$, $p = .06$).

9. General discussion

9.1. Summary

In this paper, we report a series of experiments that utilized a relational priming paradigm to test conceptual understanding and procedural proficiency with multiplication with rational numbers, and in particular to examine a potential benefit of componential processing of fractions. In a speeded multiplication task, college students judged whether sequentially presented equations using fractions were correct or incorrect. In true primed equations, some successive equations were paired such that the prime and target had an equation inverse relationship. The whole numbers that formed the prime and target might be identical (e.g., $4 \times 3/4 = 3$ followed by $3 \times 4/3 = 4$) or only relationally similar (e.g., $4 \times 3/4 = 3$ followed by $3 \times 8/6 = 4$). Experiment 1 established that for college students at a highly ranked public university, priming is obtained in true primed cases when the constituent numbers are identical. In Experiment 2, we demonstrated that the range of relational priming varies across levels of overall math ability. Low-performing participants showed a priming effect for perceptually similar prime-target pairs (identical constituent whole numbers) but not for pairs that were relationally but not perceptually similar. In contrast, high-performing participants showed transfer from prime to target even for prime-target pairs that were related relationally but not perceptually. In Experiments 3–5, we tested the limits of the priming effect. Results showed that relational priming depends on (a) structural equivalence between equations (Experiments 3B and 4) and on (b) the ability to divide constituent numbers across equations in each of the paired equations (Experiment 5). Moreover, relational priming goes beyond simple perceptual matching (Experiment 3A).

9.2. Fractions as relational representations

The results of this study suggest that fraction notation provides a componential representation of rational numbers, which can be exploited effectively by relative math experts. The relational structure inherent in the bipartite fraction format provides greater support for access to key mathematical relations, including common factors. Because division and multiplication are inverse operations, people with reasonably high math ability (e.g., college students at a highly ranked public university) are able to view a fraction as if it were simply a type of division operation, and hence exploit the multiplicative schema that links division with multiplication (Campbell, 1997, 1999a; Campbell & Alberts, 2010). As a consequence, such participants were able to capitalize on the affordances available in the fraction notation.

The relations made explicit in the fraction format aid in detecting whether there are any reducible relationships between the first factor and the divisor that would simplify an expression. For example, in the equation $12 \times 5/6 = 10$, one can either multiply 12×5 and then divide that product by 6; or conversely, divide 12 by 6 and multiply that quotient by 5. In the latter case, the left-hand-side of the equation simplifies to 2×5. Thus, there are several different ways to compute the answer to the equation. Armed with an understanding of all the relations within the equation, simple strategies for verification can be used.

9.3. Fractions as visual objects

Our results suggest that fraction notation serves as a form of perceptual grouping that highlights spatial structure in a way that goes beyond simple perceptual matching. The present findings are broadly consistent with previous work indicating that both the semantic content and the perceptual structure of mathematical formalisms can have an impact on mathematical processing (Bassok, 1996; Campbell, 1999b; Landy & Goldstone, 2007a; McNeil & Alibali, 2004). Adults with relatively high math ability likely have greater perceptual expertise with fractions, which can in turn facilitate detection of connections between alternative simplified or reduced forms of a fraction (cf. Kellman et al., 2008). In this study, priming was observed for high-ability participants as long as the formal structure "chunked" successive equations in a manner that maintained the relational similarity of the prime and target. For example, in Experiment 3A, priming was obtained when the prime was in the form $a \times b/c = d$ and the target was in the form $c/b \times d = a$, so that a key chunk (b/c and its inverse c/b) was preserved. In contrast, in Experiment 3B, priming was not obtained when the form of the prime was $a \times b/c = d$ and that of the target was $c \times d/b = a$. In this case, the relational chunk (b/c and its reciprocal c/b) was not preserved, even though formally the target did have c/b embedded within it (in the derivable form of cd/b). Facilitation was observed when these perceptual chunks were provided by the fraction notation, even though all equations are readily solvable using division.

Relational priming may arise as an interaction between reasoning and higher level visual processes such as grouping, parsing, and pattern matching. The Perceptual Manipulations Theory (PMT; Landy, Allen, & Zednik, 2014) provides a possible theoretical account of the role of perceptual processes in evaluating notational systems. From the perspective of PMT, cognition is opportunistic, employing direct perceptual routines when they are easily applicable, and otherwise manipulating sensorimotor representations corresponding to physical formalisms. For a successful learner, these perceptual routines are directly linked to mathematical concepts, rules, and procedures. Under the PMT view, an "expert" in math has learned how to perceptually and physically interact with notations in a way that corresponds with their mathematical meanings. The PMT assumes that this alignment of perceptual routines with mathematical concepts and procedures develops with substantial training in use of a notational system.

9.4. Broader implications for relational processing

The present findings support the hypothesis that fractions have an important relational component (DeWolf et al., 2015a), which college students have learned and can exploit to increase efficiency in evaluating multiplication equations. Thompson and Saldanha (2003) have argued that a high-level fraction schema, which includes understanding of reciprocal and division relationships, requires integration with a strong multiplication schema. A relational understanding of fractions relies on a relational understanding of multiplication and division, which goes beyond treating these operations simply as procedures on quantities.

The present demonstration of implicit activation of the reciprocal relation is encouraging in light of the common criticism that American math education fails to stress relational connections among mathematical concepts (e.g., Richland, Stigler, & Holyoak, 2012). It seems that at least the more successful students, who are more likely to eventually enter a highly competitive university, have in fact achieved a flexible understanding of fractions (especially in the context of multiplication and division). Such students are able to treat the whole number components of the fraction as two separate numbers being divided. Further, they are able to integrate this division relationship within the larger context of a multiplication equation with whole numbers. Thus, not only can the components of a fraction be utilized for making relative magnitude judgments, as has been shown previously (e.g., Bonato et al., 2007), but in addition these components can be used in the larger contexts of multiplication equations involving whole numbers.

The task used in this study can be viewed as a multiplicative analog to the arithmetic Piagetian inversion task (e.g., Wubbena, 2013). The priming observed in the present task is dependent on recognizing (at least implicitly) the inversion between the prime and target. The type of logic needed to identify the inverse relation across problems is similar to the manner in which children must reason about math equivalence in other domains, such as addition (Carpenter, Franke & Levi, 2003; Rittle-Johnson et al., 2011). In the present task, adults must recognize the inverse relation at a rapid pace, and are not explicitly asked about whether or not problems are equivalent. However, the observed priming provides evidence of at least some implicit recognition of equivalence across problems, especially for those that are perceptually similar.

Importantly, the priming advantage observed in this study for fractions in multiplication, and in the priming of inverse expressions, is not found universally for less-expert participants (Experiment 2), who are prone to the common fraction-related misconceptions that plague many students (e.g., Givvin, Stigler, & Thompson, 2011; Stafylidou & Vosniadou, 2004; Stigler, Givvin, & Thompson, 2010). People with less mathematical knowledge do not exhibit a flexible understanding of the relationships within a fraction, and hence may not exploit the relational properties of equations to reduce or simplify them. Furthermore, in the absence of a perceptual match, lower performing participants may be less likely to appreciate that these problems are in fact related inversely.

This study provides evidence that expertise with fractions in the context of equations that require multiplication and division depends on the ability to recognize relational similarities between numbers in terms of their common factors. Our findings suggest that the conceptual understanding of fractions should be defined in relation to the uses of fractions in particular contexts. In multiplication equations of the sort we used in our experiments, there is no need to evaluate the magnitude of the fraction as a whole, but it is advantageous to be able to manipulate the component parts of the fraction and recognize common-factor relations among fraction components and whole numbers. Even though magnitude understanding is clearly important to grasping the meaning of fractions, the present results indicate that magnitude understanding should not be considered the sole measure of general understanding of fractions. Comparing performance across measures, such as assessments of numerical similarity based on mathematical relations (Murphy et al., 2015), may provide further insight into all the skills involved in "understanding fractions."

The priming paradigm introduced here may in fact be a valuable assessment tool, serving as an indicator of whether or not students possess a fluent understanding of a fraction as a representation of a relationship (e.g., a/b is a relationship between a and b, not just a magnitude). It may be that those students who achieve an understanding of fractions as a relation will also be able to grasp other important mathematical relationships, such as those that underlie algebra. There is already some evidence suggesting an important link between fraction understanding and algebra. DeWolf et al. (2015b) found that for beginning algebra students, a measure of relational fraction understanding uniquely predicted algebra knowledge even after controlling for fraction magnitude knowledge and understanding of fraction procedures.

A survey of Algebra I teachers found that poor fraction knowledge is one of two major difficulties facing math students as they begin learning algebra (National Opinion Research Center, 2007). In addition, the National Mathematics Advisory Panel (2008) found that learning of fractions is essential for mastering algebra and more complex mathematics. Fractions have a dual status that poses particular challenges for students: a fraction is at once a relationship between two quantities and also the magnitude corresponding to the division of the numerator by the denominator. Similar dualities arise in algebra, as when students must understand that an algebraic expression such as $4a$ at once represents the operation $4 \times a$ and the product of that operation (Empson, Levi, & Carpenter, 2011; Sfard & Linchevski, 1994). Thus, fractions provide the first opportunity for students to master this concept of a dual expression.

In conclusion, this study provides evidence that students at a selective public university have achieved a flexible understanding of fractions as relational expressions. Their knowledge of fractions is integrated with knowledge of whole numbers within a multiplicative schema that links common factors with products, and division with multiplication. These adults are able to capitalize on this understanding to flexibly evaluate multiplication expressions, exhibiting priming based on the inverse relationship coupled with shared common-factor relations. By identifying behavioral signatures of a sophisticated multiplicative schema embracing multiple number types (whole numbers and fractions), this study lays the groundwork for future work examining developmental changes in this type of schema. Fractions can be usefully distinguished from other rational numbers, in that they provide a unique opportunity for students to learn important multiplicative and reciprocal relationships, helping to build the foundation for success with more advanced mathematics.

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Notes

1. Throughout this paper we use the term *equation inverse relation* (or inverse equations) to refer to pairs of problems that each contain reciprocal fractions as multipliers (i.e., a/b, b/a), and in which the whole number multiplier and the whole number product switch positions. In mathematics the term "inverse" is traditionally reserved for relations between operations. In this paper, we use the term to refer to the relation between these *pairs* of equations because both utilize reciprocals (which are multiplicative inverses) and they share an overall relational structure. In our process model and when describing the pairs of problems, we refer to the equation inverse relation to explicate how the two equations are linked.

2. This "rounding" wording was used because in Experiments 1, 2, and 5 we also included a between-subjects condition using decimal equations, matched in magnitude to the fraction equations. The corresponding "true" decimal equations most often generated a decimal that required rounding to the nearest whole number. Thus, most of the decimals were not exact reciprocals because they had been rounded. No priming was observed in any experiment for decimal equations, perhaps because of the fact that they required rounding. Of course, decimals also obscure the structural relation between inverse problems, and in fact priming was not observed even for the few decimal problems that did not require rounding. Because all the hypotheses of interest concern priming with various types of fraction problems, we do not report results for the decimal conditions.

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Supporting Information

Additional Supporting Information may be found online in the supporting information tab for this article:

Appendix S1. Multiplication stimuli.

Appendix S2. Math-assessment questions.