DeWolf, M., & Holyoak, K. J. (2014). Reciprocal and multiplicative relational reasoning with rational numbers. In B. Bello, M. Guarini, M. McShane, & B. Scassellati (Eds.), *Proceedings of the 36th Annual Conference of the Cognitive Science Society* (pp. 2139-2144). Austin, TX: Cognitive Science Society.

Reciprocal and Multiplicative Relational Reasoning with Rational Numbers

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Abstract

Developmental research has focused on the challenges that fractions pose to students in comparison to whole numbers. Usually the issues are blamed on children's failure to properly understand the magnitude of the fractional number because of its bipartite notation. However, recent research has shown that college-educated adults can capitalize on the structure of the fraction notation, performing more successfully with fractions than decimals in relational tasks, notably analogical reasoning. The present study examined whether this fraction advantage also holds in a more standard mathematical task, judging the veracity of multiplication problems. College students were asked to judge whether or not a multiplication problem involving either a fraction or decimal was correct. Some problems served as reciprocal primes for the problem that immediately followed it. Participants solved the fraction problems with higher accuracy than the decimals problems, and also showed significant relational priming with fractions. These findings indicate that adults can more easily identify relations between factors when rational numbers are expressed as fractions rather than decimals.

Keywords: Relational reasoning; number concepts; fractions; decimals; priming; mathematics education

Introduction

Mathematical and Relational Reasoning

The core of a deep conceptual understanding of mathematics is the realization that mathematics is a system of relations between quantities (Richland, Stigler & Holyoak, 2012). For example, understanding addition requires grasping the commutative property, $a + b = b + a$. This property essentially expresses an interchangeable relation between the two addends and their sum. Not only is this understanding important for the concept of addition, but more broadly it provides an important stepping stone for more complex relations, such as the commutative property in multiplication and various algebraic expressions.

Although it might seem obvious that developing a strong conceptual understanding of any topic within mathematics is important, the American education system often focuses on memorization of rote procedures (Stigler & Hiebert, 1999; Rittle-Johnson & Star, 2007). This emphasis on procedures rather than conceptual understanding may have detrimental consequences for many areas within mathematics.

Fractions as Relations

One area of particular difficulty for students involves the procedural and conceptual understanding of rational numbers, and more specifically fractions. Fractions are closely linked to relational reasoning in that fractions are themselves relational expressions. Much of the research directed at how children and adults understand fractions has focused on how people mentally represent magnitudes of fractions, and associated misconceptions about how their parts (numerators and denominators) affect their sizes (see Siegler, Fazio, Bailey & Zhou, 2013, for a review). For example, Staflyidou and Vosniadou (2004) found that middle-school students typically have a misunderstanding that the value of a fraction increases when the numerator and denominator increase; or conversely, that the value of the fraction increases as the numerator and denominator decrease. Even among highly-educated college undergraduates, there is evidence that people do not represent fraction magnitudes with the same automaticity as the magnitudes of either whole numbers or decimals (DeWolf, Grounds, Bassok & Holyoak, 2013). The multiple difficulties associated with fractions raise the question, *what are fractions good for?*

A potential answer to this question may be found in the very aspect of fractions with which students seem to struggle the most: their bipartite (*a/b*) format. This format allows fractions to represent relational expressions. Decimals, by contrast, typically represent a one-dimensional magnitude. DeWolf, Bassok and Holyoak (2013, under review) found that the relational format of fractions is actually especially helpful for reasoning about relations between numbers and the sets or quantities that they are meant to represent. Adults were better able to identify particular relationships (part-to-part ratio or part-to-whole ratio) between rational numbers and a set of visually-displayed items when the rational number was shown as a fraction rather than as a decimal. The fraction format allows for direct mapping between numbers within the fraction (numerator and denominator) and the sets or quantities they are meant to represent.

It thus seems that fractions are useful for understanding relations between values and quantities. They may also be useful in understanding relations between numbers themselves, especially when used in complex expressions.

For example, specific multiplicative and division relationships exist between whole numbers. Children learn about these relationships before they learn about fractions, when they are memorizing their multiplication and division tables. For example, children learn early on that any even number is divisible by 2; and conversely, that multiplying any number by 2 results in an even number.

Fractions often utilize and highlight these relationships. For example, we know that 6/8 can be reduced because both the numerator and denominator are divisible by 2, and therefore we can express the same relationship with an equivalent fraction (3/4). Conversely, we know that 6/7 is not reducible because the numerator and denominator do not share any common factors.

Because fractions inherently express a division relationship, using them when computing a multiplication or division problem involves understanding not only the relations within the multiplication and division problem itself, but also how these relate to the relations within the fraction. Over the course of learning, students and teachers often seem to have trouble with multiplication and division with fractions. Students may get confused because of the differing procedures between fraction multiplication and fraction addition—in the latter case one needs to find a common denominator, but in the former case this is not necessary (Siegler et al., 2011, 2013). Even community-college students seem to have difficulty determining whether to obtain common denominators or multiply each number individually (Stigler et al., 2010). Similarly, a well-known procedure for fraction division is the "invert and multiply" technique, in which one keeps the first factor, changes the division sign to a multiplication sign, and inverts the second factor. Even prospective teachers show little understanding of this strategy and have difficulty explaining why this technique works (Tirosh, 2000).

At the essence of all of these errors is a lack of true conceptual understanding of multiplication. Such understanding goes beyond simply conceptualizing multiplication as repeated addition (which does not fit well with the differences in procedures across arithmetic and multiplication with fractions). Thompson and Saldanha (2003) proposed that understanding multiplication conceptually requires grasping multiple reciprocal relationships between the two factors, n and m, and their product nm (e.g., nm is n times as large as m; or conversely, n is $1/m$ times as large as nm). Thus, multiplication with fractions involves a relationally rich context.

Multiplication with decimals, the magnitude equivalents of fractions, is usually taught as a more natural extension of multiplication with whole numbers. The procedures for decimal multiplication are essentially identical to whole-number multiplication, with the extra aspect that place value in the final product is based on the number of decimal digits in the two factors (Hiebert & Wearne, 1985). Over the course of learning, students at times struggle with assessing this place value, but in general seem to do better in adapting the procedures for decimal multiplication as compared to fraction multiplication (Behr & Post, 1992).

The main goal of the present study was to assess whether college-educated adults demonstrate conceptually rich understanding of multiplication with rational numbers. Given its closer linkage to multiplication with whole numbers, it would be expected that multiplication with decimals should be easier (or at least no more difficult) than multiplication with fractions. However, if highly-educated adults show a deeper conceptual understanding of the important relations between the division and multiplication relationships involved in fraction multiplication, then fraction multiplication might actually have an advantage over decimal multiplication.

Relational Priming

To provide a sensitive measure of possible differences in performance between fraction and decimal multiplication for highly-educated adults, we adapted a relational priming paradigm previously employed with verbal relations (Spellman, Holyoak & Morrison, 2001) to create an implicit measure of whether adults use the relationships within multiplication equations to their advantage. In a product-verification task, we primed certain multiplication equations by first showing their reciprocal equation. For example, the equation 8 X 12/8 = 12 was preceded by the equation 12 X 8/12 = 8. The two fractions used are reciprocals, and the equations themselves are reciprocally related. If participants were able to isolate the important relationships in the first equation, then relations might be primed for use in determining whether the second equation is correct. For comparison, we also used decimals in place of the fractions in the multiplication problems. We thus tested whether fraction notation better highlights reciprocal relationships compared to a different type of notation, decimal numbers. In Experiment 1, we tested the difference in priming when problems with presented with fractions compared to decimals. In Experiment 2, we examined whether this difference in priming would also hold when the equivalent fractions do not have identical whole number parts to those numbers in the rest of the expression (e.g., 8 X 3/2 = 12, rather than 8 X 12/8 = 12).

Experiment 1

Method

Participants Participants were 60 undergraduates from the University of California, Los Angeles (UCLA) (mean age = 20; 47 females) who received course credit. Thirty participants were randomly assigned to two between-subjects conditions.

Design and Materials The study was a 2 (number type: fractions vs. decimals) X 2 (trial type: first trial vs. primed trial) X 2 (problem type: true vs. false) design, with number type as a between-subjects factor and trial type and problem type as within-subjects factors.

The stimuli were all multiplication problems with the form *A* X *B/D* = *C*, where *B/D* was either shown as a fraction or as its equivalent decimal rounded to two decimal places (e.g., 12 X 1/6 = 2 or 12 X .17 = 2). There were a total of 240 problems, half of which were correct.

Half of the trials consisted of prime pairs (a prime followed by its primed reciprocal). Correct primed pairs were of the form *A* X *B/A* = *B* and *B* X *A/B* = *A*, where *B/A* and *A/B* share a reciprocal relationship. For such pairs, if the participant correctly identifies the first trial in the pair as being true, and implicitly or explicitly recognizes the reciprocal relationship between the two pairs, then the second problem can be solved without needing to do any calculation to verify the product. Incorrect primed pairs followed the same superficial relationship, but the product was incorrect for both trials (e.g., *A* X *B/A* = *C* and *C* X *A/B* = *A*). The same reciprocal relationship holds between *B/A* and *A/B*, but the internal multiplicative relationship within each problem does not hold. The order of appearance of the trials within each primed pair was random, so that some participants were primed with the *A/B* version of the equation and some were primed with the *B/A* version.

The unprimed trials included a variety of foils and fillers. The true fillers followed the same form as the true prime trials, but some used other equivalent fractions as the second multiplier for variety (e.g., 8 X 3/2 = 12). In addition, false trials included perceptual foils related to the true trials (e.g., *A* X *A/B* = *B* instead of *A* X *B/A* = *B*). As with the prime trials, half of the unprimed trials were correct. Problem order was varied so that participants saw true trials followed by true trials, false trials followed by false trials, and true trials followed by false trials and vice versa. Accordingly, any priming effect could not be attributed to a general bias to hit the same key twice in succession, but rather could be attributed to the relational similarity between the primed trial and its preceding trial.

Procedure Stimuli were displayed with Macintosh computers using Superlab 4.5, and response times and accuracy were recorded. Participants were instructed that they would need to decide whether a series of multiplication problems were correct or incorrect. If the problem was correct, they were instructed to hit the *a* key; if it was incorrect they were to hit the *l* key. Participants were told that the answers were shown rounded to the nearest whole number. As we were particularly interested in potentially subtle response time differences, participants were instructed to respond as quickly as possible while maintaining high accuracy. There was no time limit for responding. They were first given four practice trials that used only whole numbers. After the practice trials, they were given a chance to ask remaining questions before starting the test trials.

Results

Accuracy Across all problem types, participants were more accurate for fraction problems than decimal problems (90% vs. 78%; $t(58) = 5.30$, $p < .001$). Accuracy for each participant was averaged for each trial of each of the true and false primed pairs. Mean accuracy values for prime pairs (i.e., the prime problem on trial 1 and the primed problem on trial 2) are shown in Figure 1. A 2 (number type) X 2 (trial type) X 2 (problem type) mixed factors ANOVA yielded a strong effect of number type favoring fractions over decimals (94% vs. 78%; $F(1, 58) = 41.19$, $p < .001$). There was no effect of trial type ($F(1, 58) = .117$, $p = .733$), indicating there was no general priming effect on accuracy. Also, there was no significant interaction between number type and trial type ($F(1, 58) = .094$, $p = .761$), and thus no differential priming effect for fractions over decimals in accuracy; nor was there a reliable 3-way interaction ($F(1, 58) = .093$, $p = .762$).

Response Times Across all problem types, response times for fraction problems were faster than response times for decimal problems (2.76 s vs. 4.03 s; $t(58) = 3.93$, $p = .001$). Response times for each participant were averaged over each trial for each of the true and false primed pairs. Response times for incorrect answers were excluded from the analyses. Mean response times are shown in Figure 2. A 2 (number type) X 2 (trial type) X 2 (problem type) mixed factors ANOVA yielded a significant 3-way interaction ($F(1, 58) = 8.27$, $p = .006$). There was also a significant 2-way interaction for the true primed pairs ($F(1, 58) = 10.72$,

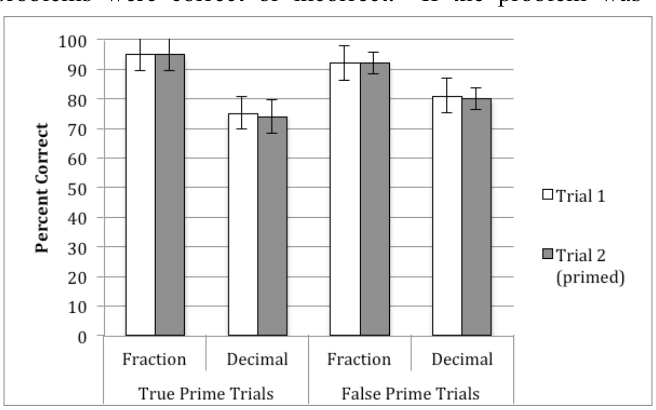

Figure 1. Percent correct for pairs of prime trials for true and false expressions, separated by number type (fraction or decimal) for Experiment 1.

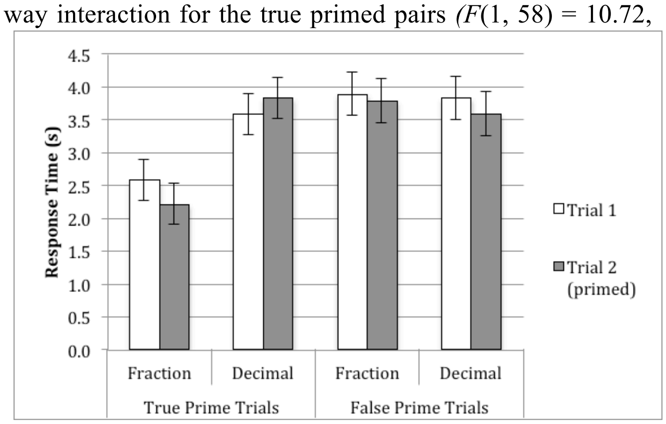

Figure 2. Response times for pairs of prime trials for true and false expressions, separated by number type (fraction or decimal) for Experiment 1.

p = .002), indicating a differential priming effect for fractions compared to decimals. Planned comparisons revealed that for fraction problems, the second (primed) trial was significantly faster than the first (prime) trial (2.21 s vs. 2.58 s; $t(1, 29) = 3.08$, $p = .004$). By contrast, the decimal problems showed no significant difference for the primed trial relative to the priming trial (3.84 s vs. 3.58 s $t(29) = 1.72$, $p = .10$). Thus, response times show a significant speed-up consistent with a priming effect for fractions, but revealed no reliable priming effect for decimals.

For the false prime trials, there was no effect of trial type ($F(1, 58) = 3.158$, $p = .081$) or number type ($F(1, 58) = .08$, $p = .778$), nor a significant 2-way interaction ($F(1, 58) = .537$; $p = .467$). Thus, the priming effect on response times for fractions was only observed when the problems were correct.

Experiment 2

Method

Participants Participants were 87 UCLA undergraduates (mean age = 20; 70 females) who received course credit. Twenty-nine participants were randomly assigned to three between-subjects conditions.

Design, Materials and Procedure The design and materials were the same as used in Experiment 1, with the exception that an additional between-subjects condition was added. This additional fraction condition contained true prime problems that were identical to the original fractions problems except that the fraction did not match one-to-one with the other whole numbers in the problem. We term this condition "non-matching" fractions. The original fraction condition from Experiment 1 will be referred to as the "matching" fractions condition. For example, a problem in the matching fractions condition would be 12 X 8/12 = 8; for non-matching fractions, the corresponding problem would be 12 X 4/6 = 8; for decimals, 12 X .67 = 8. The fractions in the non-matching condition were created by either multiplying the original fractions by 2/2 or 3/3 or by reducing the fractions by the same factors. Because of these additional constraints, the set of problems used in Experiment 2 was slightly different from that used than Experiment 1, in order to increase the number of fraction problems in which equivalent fractions were possible to use. The non-matching and matching fractions conditions were identical for the foil problems and the false prime problems. The foil problems and false prime problems were identical to those used in Experiment 1. The procedure was also identical.

Results

Accuracy Across all problem types, participants were more accurate for the two types of fraction problems than for decimal problems ($F(2, 84) = 8.17$, $p = .001$). Planned comparisons showed that matching fraction accuracy was greater than decimal accuracy (91% vs. 83%; $t(56) = 3.20$, $p = .002$), and non-matching fraction accuracy was also greater than decimal accuracy (90% vs. 83%; $t(56) = 3.10$, $p = .003$). There was no difference between matching fraction and non-matching fraction accuracy (91% vs. 90%; $t(56) = .42$, $p = .68$).

As in Experiment 1, accuracy for each participant was averaged for each trial of each of the true and false primed pairs. Mean accuracy values for prime pairs are shown in Figure 3. A 3 (number type) X 2 (trial type) X 2 (problem type) mixed factors ANOVA yielded a significant 3-way interaction ($F(1, 84) = 3.685$, $p = .029$). There was also a significant 2-way interaction for true prime pairs ($F(1, 84) = 3.90$, $p = .024$), indicating a differential priming effect. The non-matching fractions showed a significant increase in accuracy from the first trial to the second primed trial (85% vs. 90%; $F(1, 84) = 9.17$, $p = .003$). There was no difference in accuracy between primes and primed trials for matching fraction problems (94% vs. 94%; $F(1, 84) = .264$, $p = .61$) or decimal problems (85% vs. 83%; $F(1, 84) = .756$, $p = .39$).

For false prime trials, there was no interaction between trial type and number type ($F(1, 84) = .832$, $p = .44$), indicating no differential priming effect between fractions and decimals. There was no main effect of trial type ($F(1, 84) = .018$, $p = .89$), implying no increase in accuracy from the first trial to the primed trial across number types. There was, however, a significant effect of number type ($F(1, 84) = 13.57$, $p < .001$), which followed the same pattern as for overall accuracy: there was no difference in accuracy for matching fractions and non-matching fractions (93% vs. 94%; $t(56) = .74$, $p = .47$), but both matching fractions and non-matching fractions had higher accuracy than decimals (93% vs. 82%; $t(56) = 3.601$, $p = .001$; 94% vs. 82%; $t(56) = 4.335$, $p < .001$).

Response Time Across all problem types, response times for matching and non-matching fraction problems were faster than those for decimal problems ($F(2, 84) = 5.140$, $p = .008$). Decimals were slower than matching fractions (4.42 s vs. 3.22 s; $t(56) = 2.72$, $p = .009$) and non-matching fractions (4.42 s vs. 3.42 s; $t(56) = 2.37$, $p = .021$). There was no difference in response time for matching and non-matching fractions (3.22 s vs. 3.42 s; $t(56) = .596$, $p = .55$). Figure 4 shows the response times for true prime and false prime trials by number type. A 3 (number type) X 2 (trial type) X 2 (problem type) mixed factors ANOVA did not yield a significant 3-way interaction ($F(1, 84) = .227$, $p = .797$). There was, however, a significant differential priming effect, as indicated by a 2-way interaction between trial type and number type ($F(1, 84) = 3.76$, $p = .027$). Decimals did not show a significant decrease in response

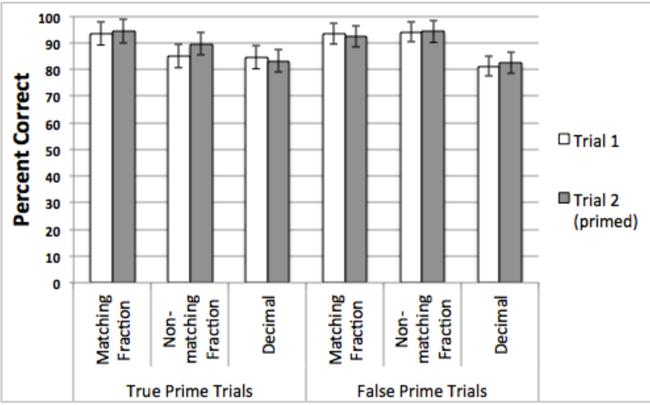

Figure 3. Percent correct for pairs of prime trials for true and false expressions, separated by number type (matching fraction, non-matching fraction, or decimal) for Experiment 2.

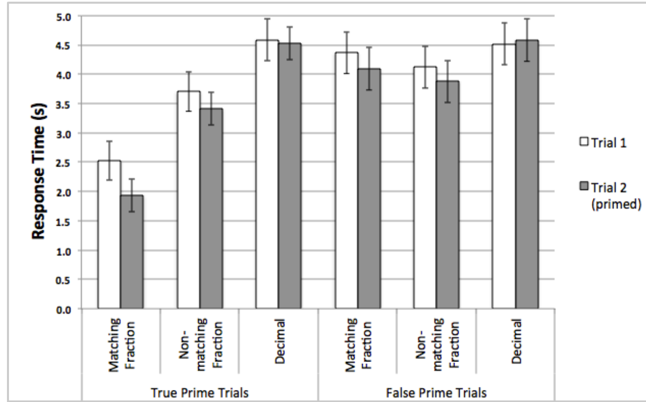

Figure 4. Response times for pairs of prime trials for true and false expressions, separated by number type (matching fraction, non-matching fraction, or decimal) for Experiment 2.

time from the first trial to the second primed trial (4.55 s vs. 4.55 s; $F(1, 84) = 0$, ns.). However, both matching fractions (3.45 s vs. 3.01 s; $F(1, 84) = 14.73$, $p < .001$) and non-matching fractions (3.91 s vs. 3.64 s; $F(1, 84) = 5.58$, $p = .02$) showed a significant decrease in response time.

Relative to Experiment 1, fractions showed a slightly stronger priming effect in Experiment 2. In Experiment 1, the true primed trials for the decimal condition showed a trend towards increased response time (negative priming); however, no such trend was observed in Experiment 2. Most importantly, priming of reciprocal relations was observed for non-matching as well as matching fractions, but not for decimals.

Discussion

Our results demonstrate an advantage for fractions over decimals in verifying whether a multiplication problem is correct, and in implicitly identifying reciprocal relations between problems. College students were considerably faster and more accurate when judging whether multiplication problems with fractions were correct, compared to otherwise identical problems with decimals. This result highlights the importance of the relational structure of a fraction. Because division and multiplication are inverse operations, highly-educated people are able to view a fraction as if it were simply a type of division operation. This ability then enables them to detect whether there are any reducible relationships between the first factor and the divisor to simplify the expression. For example, in the problem 12 X 5/6 = 10, one can either multiply 12 X 5 and then divide that product by 6; or conversely, divide 12 by 6 and multiply that quotient by 5. In the latter case, the problem simplifies to 2 X 5. Thus, there are several different ways to compute the equation. With a flexible and deep understanding of the relations within the problem, very simple strategies for verification can be used. In the case of decimal multiplication, there is no corresponding way to simplify the problem. In the corresponding example, 12 X .83 = 10, one must either estimate the answer, perform the full double-digit multiplication, or break down the equation into something a little simpler, such as 10 X .83 + 2 X .83. Any of these strategies will most likely be less precise or take somewhat longer than the optimal fraction strategy.

The finding that there was a significant speed-up in solving primed fraction problems, but no parallel speed-up in solving primed decimal problems, also provides evidence that adults possess a deep understanding of the reciprocal relationship between fractions. Whereas reciprocals are superficially obvious with fractions (4/7 vs. 7/4), decimal reciprocals are much less apparent (.57 vs. 1.75). This superficial similarity may have contributed in part to the priming in Experiment 1. But even in Experiment 2, where the non-matching fractions were less transparently related to the whole numbers in the problems, participants still recognized and exploited the reciprocal relation between problems. Beyond this, participants showed a deeper understanding of what the reciprocal relation means when represented with fractions. They apparently recognized that corresponding expressions were mathematically equivalent and therefore did not require additional verification. For fraction problems, encountering the two reciprocal equations in succession provided an easily recognizable hint for participants, whereas the same juxtaposition of decimal problems did not provide any help.

In summary, the present findings support the hypothesis that fractions have an important relational component, which college students have learned and use to increase efficiency in evaluating multiplication problem. Thompson and Saldanha (2003) have argued that a high-level fraction schema, which includes understanding of reciprocal and division relationships, requires integration with a strong multiplication schema. That is, because a fraction is inherently a relational expression, important operational relations such as multiplication and division must be understood. This relational knowledge allows recognition of numbers that are known to share common factors, or that can be simplified or reduced to make a computation easier.

Recent research on the acquisition of fractions has generally aimed to understand why children and even adults have trouble integrating their knowledge about fractions with their knowledge about whole numbers. Focus on this issue has perhaps contributed to a failure to appreciate that fractions are inherently different from other types of numbers (even other rational numbers, such as decimals). Fractions have many different meanings (e.g., part/whole, ratio, proportion, subset/set, or quotient). It may be useful to evaluate whether children understand the relational importance of fractions in other contexts beyond magnitudes (Stafylidou & Vosniadou, 2004).

In addition, when fractions are conceptualized as relational expressions, they become a stepping-stone to algebra. There is already some evidence suggesting an important link between algebra and fraction understanding. A survey of Algebra I teachers found that poor fraction knowledge is one of two major difficulties facing math students as they begin learning algebra (NORC, 2008). In addition, the National Mathematics Advisory Panel (2008) found that learning of fractions is essential for mastering algebra and more complex mathematics. Fractions have a dual status that poses particular challenges for students: a fraction is at once a relationship between two quantities and also the magnitude corresponding to the division of the numerator by the denominator. Similar dualities arise in algebra, as when students must understand that an algebraic expression such as 4*a* at once represents the operation 4 X *a* and the product of that operation (Sfard & Linchevski, 1994; Empson, Levi, & Carpenter, 2011). Thus, fractions provide the first opportunity for students to master this concept of a dual expression.

In summary, the current study provides evidence that highly-educated adults demonstrate a flexible understanding of fractions as relational expressions. They are able to capitalize on this understanding in order to solve multiplication verification problems, exhibiting priming based on the reciprocal relationship. Fractions, then, can be usefully distinguished from other rational numbers, in that they provide a unique opportunity for students to learn important multiplicative and reciprocal relationships.

Acknowledgments

Preparation of this paper was supported by NSF Fellowship DGE-1144087 and grant number R305C080015 from the Institute of Education Sciences. We thank Austin Chau, Queenie Cui, Eugene Goh, Andrew Molica, and Marvin Lopez for help with collecting and analyzing data.

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